Numerical approximation of conservation laws with non-local and discontinuous fluxes

a thesis submitted for the award of the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

by

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October 2025

Declaration

I, **Nikhil Manoj (Roll No: PHD201033)**, hereby declare that, this thesis entitled **"Numerical approximation of conservation laws with non-local and discontinuous fluxes"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards the partial fulfilment of the requirement for the award of the degree **Doctor of Philosophy** in **School of Mathematics**, is an original work carried out by me under the supervision of **Dr. Sudarshan Kumar K.** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold academic ethics and honesty. Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.

Thiruvananthapuram - 695551 **Nikhil Manoj**

October 17, 2025

Certificate

This is to certify that the work contained in this project report entitled **"Numerical approximation of conservation laws with non-local and discontinuous fluxes"** submitted by **Nikhil Manoj** (**Roll No: PHD201033**) to Indian Institute of Science Education and Research Thiruvananthapuram towards the partial fulfilment of the requirement for the award of the degree **Doctor of Philosophy** in **School of Mathematics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695551 | **Dr. Sudarshan Kumar K.**

October 17, 2025 | Thesis Supervisor

Acknowledgements

I would like to express my deepest gratitude to my thesis supervisor, Dr. Sudarshan, for his unwavering support and encouragement over the past four years. His enthusiasm and ever-energetic presence made this journey truly enjoyable. He was always available for a discussion and willing to go the extra mile. I will always cherish the excitement he radiated while discussing mathematics, his keen eye for detail, and his proactive nature, all of which have left a lasting impression on me. I am especially grateful for the care and mental support he extended during my recovery from an injury. I thank him for patiently guiding me through the first steps of my research and for his insightful advice throughout.

It has been an honour to work with Prof. Veerappa Gowda. His warmth and words of encouragement meant a great deal to me as a young researcher. I have deep admiration for his erudition, and for his passion and excitement for research and teaching.

I extend my sincere thanks to my doctoral committee members, Dr. Arun K.R. and Dr. Nagaiah Chamakuri, for their valuable suggestions and feedback throughout these years. I am also grateful to all the faculty members in the department, especially Dr. Asha Dond, Dr. Sheetal Dharmatti, Dr. Viji Thomas, and Dr. Sachindranath Jayaraman. Special thanks to Dr. Dhanya Rajendran for her thoughtful suggestions and encouragement at various stages of my Ph.D.. I owe much to some wonderful teachers who guided me early on. I am especially thankful to Dr. Shine C. Mathew from St. Berchmans College for introducing me to the elegant world of higher mathematics. I also extend my thanks to Dr. Kiran Kumar V.B. from Cochin University for his inspiring lectures during my M.Sc. days.

I sincerely thank IISER Thiruvananthapuram for providing a supportive research environment and excellent infrastructure, and for offering financial assistance for my conference travel. I also gratefully acknowledge the Council of Scientific and Industrial Research, Government of India, for the doctoral fellowship.

My time at IISER was made rich and memorable by the companionship of many lovely people. I thank Samrat, Tony, Sohela, Dimpy, John, Aishwarya, Swetha, Aswathy, Reshma, Naseem, Vishnulal, Jervis, Radhikesh, Niloopher, Pauly, Shreyas and Diksha. I am also grateful to my colleagues from the Math department- Subham, Muneeswaran, Ritabrata, Manika, Rahuldev, Shrinath, Pallavi, Himanshi, Greeshma, Amogh, Tooba, Mainak, Sulakhana, Nanditha, Mahendra and Soham. Thanks also to Bichu, Sahad, Raees, Jibin, Renitta, and Alvin. I extend my gratitude to my longtime friends Rahul, Sachin, Amaya, Kavya, Anagha, Albin, Shivanand and Ashin for their enduring friendship.

If it were not for the strong and loving people who make up my family, I would not have come this far. I remember my late father, Manoj Kumar K., who introduced me to the world of books and mathematics during my childhood. I remain grateful for the love and care he gave me throughout his life, until his untimely passing. I thank my mother, Sunila Manoj, for raising me and my sister through uncertain times with a quiet optimism and selfless love. Lots of love to my sister and lifelong companion—Malu. I am also thankful to Chitta and Mummy for their love and support. Thanks to Subash uncle and Poornima aunty for their constant goodwill. Finally, I would like to thank my life partner and best friend, Anindita, for being beside me throughout the last four years - for keeping me grounded and sane, and for her unconditional love.

Dedication

This thesis is dedicated to my amma, Sunila Manoj.

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Abstract

This thesis focuses on the design and analysis of numerical discretizations for two classes of conservation laws with spatially dependent flux function: (i) conservation laws with non-local flux, where the flux function depends on the convolution of the unknown with a given kernel function, and (ii) conservation laws with flux function allowed to be discontinuous in the spatial variable. While robust first-order finite volume methods exist for these problems, they are often overly diffusive, motivating the development of second-and-higher-order schemes supported by rigorous theoretical results. For two major classes of one-dimensional non-local conservation laws, we develop second-order schemes based on MUSCL-type spatial reconstructions coupled with multi-stage time-stepping methods such as Runge–Kutta, and single-stage techniques like MUSCL–Hancock. The convergence analysis of these schemes is carried out within the framework of Kolmogorov's compactness theorem, by deriving a series of a priori estimates, including L^∞ and bounded variation bounds, followed by mesh-dependent modifications to ensure entropy convergence. For multi-dimensional systems of non-local conservation laws, we design a fully discrete second-order method and analytically establish that it is positivity-preserving and L^∞ stable. Finally, for a broad class of conservation laws with discontinuous flux, we propose and analyze a second-order central scheme. To address the challenges arising in the convergence analysis from the lack of bounded variation estimates and the non-monotonicity of second-order schemes, we employ the theory of compensated compactness. A mesh-dependent modification in the slope limiter and an estimate weaker than the classical bounded variation bound are employed to generalize a classical entropy convergence theory to the setting of discontinuous fluxes. The effectiveness and accuracy of the proposed schemes are demonstrated through a series of numerical experiments in comparison with existing first-order methods.

1

Introduction

Hyperbolic conservation laws constitute one of the most fundamental classes of partial differential equations, with widespread applications across diverse fields such as aerodynamics, oceanography, plasma physics, traffic flow, crowd dynamics, meteorology, flow in heterogeneous media etc. A general system of conservation laws in d space dimensions is given by (see [73, 99])

$$\partial_t \boldsymbol{\rho} + \sum_{j=1}^{d} \partial_{x_j} f_j(t, \boldsymbol{x}, \boldsymbol{\rho}) = 0, \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^n, t > 0, \tag{1.0.1}$$

where $\boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \cdot \\ \cdot \\ \rho_p \end{bmatrix}$ is the *conserved quantity,* a vector valued function from $[0, \infty) \times \mathbb{R}^d$ into $\Omega \subseteq \mathbb{R}^p$. For $j \in \{1, 2, \ldots, d\}$ the function $f_j : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^p$, written in the form $f_j = \begin{bmatrix} f_{1,j} \\ \cdot \\ \cdot \\ f_{p,j} \end{bmatrix}$ is called the *flux function* in the j-th dimension. Integrating (1.0.1) over a domain $D \subset \mathbb{R}^d$, we obtain

$$\partial_t \int_D \boldsymbol{\rho} \, \mathrm{d}\boldsymbol{x} + \sum_{j=1}^{d} \int_{\partial D} f_j(t, \boldsymbol{x}, \boldsymbol{\rho}) \cdot n_j \, \mathrm{d}S = 0, \tag{1.0.2}$$

where $\boldsymbol{n} = (n_1, n_2, \dots, n_d)$ is the outward unit normal to the boundary ∂D of D. The expression (1.0.2) indicates that the time variation of the quantity $\int_D \boldsymbol{\rho} \, \mathrm{d}\boldsymbol{x}$ happens only due to the flow across the boundary ∂D.

A fundamental problem in the theory of conservation laws is to solve the corresponding *Cauchy problem.* For a given initial datum $\boldsymbol{\rho}_0 : \mathbb{R}^d \to \Omega$, the Cauchy problem for the system (1.0.1) is to find a function $\boldsymbol{\rho} : [0, \infty) \times \mathbb{R}^d \to \Omega$ which solves (1.0.1) and satisfies the initial datum

$$\boldsymbol{\rho}(0, \boldsymbol{x}) = \boldsymbol{\rho}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

In the case when the flux functions depend only on $\boldsymbol{\rho}$, i.e. $f_j(t, \boldsymbol{x}, \boldsymbol{\rho}) = f_j(\boldsymbol{\rho})$, for $j = 1, \dots, d$, the Jacobian matrix of $\boldsymbol{f_j}$ is given by

$$\boldsymbol{A}_j(\boldsymbol{\rho}) := \left(\frac{\partial f_{ij}}{\partial \rho_k}(\boldsymbol{\rho}) \right)_{1 \le i,k \le p}.$$

The system (1.0.1) is called *hyperbolic* if for any $\boldsymbol{\rho} \in \Omega$ and any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d, \boldsymbol{\omega} \neq 0$, the matrix

$$\boldsymbol{A}(\boldsymbol{\rho}, \boldsymbol{\omega}) := \sum_{j=1}^{d} \omega_j \boldsymbol{A}_j(\boldsymbol{\rho})$$

admits p real eigen values $\lambda_1(\boldsymbol{\rho}, \boldsymbol{\omega}) \le \lambda_2(\boldsymbol{\rho}, \boldsymbol{\omega}) \le \cdots \le \lambda_p(\boldsymbol{\rho}, \boldsymbol{\omega})$ and corresponding p linearly independent eigenvectors $\boldsymbol{r}_1(\boldsymbol{\rho}, \boldsymbol{\omega}), \boldsymbol{r}_2(\boldsymbol{\rho}, \boldsymbol{\omega}), \dots, \boldsymbol{r}_p(\boldsymbol{\rho}, \boldsymbol{\omega})$. If all the eigen values are distinct, the system is called *strictly hyperbolic.*

A characteristic feature of nonlinear conservation laws is the formation of discontinuities (shocks) in finite time, even when the initial data is smooth. Consequently, classical (differentiable) solutions cease to exist beyond a finite time, and solutions must be interpreted in a weaker sense, as *weak solutions.* However, weak solutions are not necessarily unique, necessitating the imposition of an additional admissibility criterion to isolate the physically relevant solution. A foundational result in this direction is the entropy condition proposed by Kružkov [124], which is based on the vanishing viscosity method and ensures uniqueness for scalar conservation laws. For a detailed exposition on the well-posedness theory of scalar conservation laws, we refer to [124]. In the case of general systems of conservation laws, global well-posedness remains largely unresolved, with only partial results available in specific cases; see [95, 130]. An alternative framework for addressing issues of existence and uniqueness is provided by the theory of measure-valued solutions, as developed in [80, 35].

In practical applications, exact solutions to conservation laws are rarely available, thereby motivating the development of efficient and accurate numerical methods for their approximation. Moreover, numerical methods frequently play a central role in the theoretical analysis of conservation laws, particularly in establishing the existence of solutions. Classical existence results, such as those in [95, 126], are often built upon the convergence of suitably designed numerical approximations. Several milestone contributions have shaped the development of numerical methods for conservation laws. Among the earliest are the Lax–Friedrichs scheme [129] and the Godunov scheme [101], followed by the Engquist–Osher scheme [83] and the Roe scheme [146]. By the 1980s, a rigorous analytical framework for the convergence of first-order schemes had been established, notably in the work of Crandall and Majda [71]. Subsequent efforts turned toward achieving higher-order accuracy. Significant progress in this direction was made through the development of MUSCL (Monotonic Upstream-centered Scheme for Conservation Laws) schemes [161, 162, 163, 139]. Other influential approaches include the modified flux method of Harten [107], the flux limiter method of Sweby [155], the piecewise parabolic method of Colella and Woodward [170], the ENO (Essentially Non-Oscillatory) schemes of Shu and Osher [151], and the second-order central scheme introduced by Nessyahu and Tadmor [137], among others. Detailed discussions on the analytical and numerical aspects of conservation laws can be found in the standard textbooks [100, 99, 73, 36].

Over the past few decades, conservation laws with non-standard flux functions have emerged in a wide range of physical models. This thesis investigates numerical discretization and its analyis for two important classes of such equations:

(i) Conservation laws with non-local flux, where the flux function depends on a convolution of the unknown with a given kernel function,

(ii) Conservation laws with flux functions allowed to be discontinuous in the spatial variables.

Non-local fluxes account for interactions over spatial neighborhoods and arise in applications such as traffic flow, crowd dynamics, sedimentation, etc. On the other hand, discontinuous fluxes model heterogeneities in flow media, with relevance to multiphase flows, sedimentation, traffic over variable road conditions etc. In the following sections, we provide a brief survey of the theoretical and numerical developments related to these equations. Before proceeding, we introduce certain necessary notations that will be used throughout this thesis.

1.0.0.1 Notations

For two domains Ω_1 and Ω_2, the space $C^1(\Omega_1;\Omega_2)$ denotes the set of all continuously differentiable functions from Ω_1 to Ω_2. The space of all functions in $C^1(\Omega_1;\Omega_2)$ with compact support in Ω_1 are denoted by $C_c^1(\Omega_1;\Omega_2)$. The space $\mathcal{D}(\Omega)$ denotes the set of all test functions on Ω, i.e., infinitely differentiable functions with compact support in Ω.

We denote $\mathbb{R}_+ := [0, \infty)$. For a function g defined on a domain Ω, the notation $\mathrm{TV}(g)$ refers to the total variation semi-norm of g over Ω and $\mathrm{supp}(g)$ denotes the support of g. The space $\mathrm{BV}(\Omega)$ denotes the set of all functions of bounded variation on Ω. For $a, b \in \mathbb{R}$, denote $\mathcal{I}(a, b) := (\min(a, b), \max(a, b))$ and $\|\cdot\| := \|\cdot\|_{\mathrm{L}^\infty}$.

1.1 Non-local conservation laws

Conservation laws with non-local flux have emerged as powerful tools for modeling a wide range of flow-like physical processes that involve interactions over spatial neighborhoods. In contrast to the standard conservation laws, non-local models incorporate surrounding information, typically through a convolution of the unknown with a given kernel function which is included in the flux function. This formulation allows a more realistic description of physical scenarios influenced by neighborhood effects. The non-locality in the flux is particularly useful in describing real-world phenomena such as traffic flow, where a driver's response is affected by the density of cars ahead [28, 34, 53, 58, 59, 152]; crowd dynamics, where individuals respond to nearby pedestrians [40, 67, 68, 69]; and sedimentation, where the motion of particles depends on neighboring concentrations [33]. Other examples include applications in supply chains [23], conveyor belts [102, 147], weakly coupled oscillators [18], and biological population models [142].

The mathematical study of non-local conservation laws has seen steady growth in recent years. In this context, the central question of well-posedness has been addressed in numerous works (see [14, 19, 34, 47, 53, 61, 74, 96, 147], and references therein). A common approach to establish well-posedness involves designing first-order numerical schemes tailored to the non-local flux, which are then shown to converge along a subsequence to an entropy weak solution, thereby establishing existence. In the next step, continuous dependence on the data, and subsequently the uniqueness are established using a doubling of variables argument suitably adapted to the non-local setting (see [124, 33, 34, 54]). Alternate approaches for establishing well-posedness include the fixed-point argument of [119, 70]. Non-local problems have also been investigated from several other perspectives. For instance, stability estimates have been derived in [56, 109], and regularity results for solutions of these equations have been investigated in [30]. Furthermore, traveling wave solutions for non-local traffic flow models were discussed in [145, 150]. Additional extensions include studies on nonlocal balance laws [1, 119, 121] and pair-interaction models [81, 85], among others. Another important theoretical direction is the asymptotic limit problem, concerning the convergence of non-local models to their local counterparts as the convolution parameter tends to zero. This problem has been explored in several works, including [39, 38, 60, 62, 64, 63, 65, 120].

Next, we describe the mathematical details of the classes of non-local conservation laws considered in this thesis, beginning with the one-dimensional scalar equations.

1.1.1 One-dimensional scalar case

We consider non-local conservation laws of the form :

$$\begin{aligned} \partial_t \rho + \partial_x f(\rho, A(t,x)) &= 0, \qquad && (t,x) \in (0,\infty)\times\mathbb{R}, \\ \rho(0,x) &= \rho_0(x), && x \in \mathbb{R}, \end{aligned} \tag{1.1.1}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\rho : (0,\infty)\times\mathbb{R} \to \mathbb{R}$, the given flux function and the unknown quantity, respectively. The convolution term $A(t,x)$ is defined as

$$A(t,x) := \mu * \rho(t,x) = \int_{-\infty}^{\infty} \mu(x-y)\rho(t,y)\,\mathrm{d}y. \tag{1.1.2}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ denotes the convolution kernel. Solutions to (1.1.1) are interpreted in the following weak sense (see [33, 19, 14, 34]):

Definition 1.1.1. (Weak solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T)\times\mathbb{R};\mathbb{R})$, $T > 0$, is a weak solution of (1.1.1) if

$$\int_0^T \int_{-\infty}^{+\infty} \big(\rho\partial_t\varphi + f(\rho, A(t,x))\partial_x\varphi\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} \rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0 \tag{1.1.3}$$

for all $\varphi \in \mathrm{C}^1_c([0,T)\times\mathbb{R};\mathbb{R})$.

Further, an entropy solution to the problem (1.1.1) is defined (see [33, 34]) using a Kružkov type inequality adapted to the non-local scenario, as follows:

Definition 1.1.2. (Entropy solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T)\times\mathbb{R};\mathbb{R})$, $T > 0$, is an entropy weak solution of (1.1.1) if

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \Big(|\rho-\kappa|\partial_t\varphi + \mathrm{sgn}(\rho-\kappa)\big(f(\rho, A(t,x)) - f(\kappa, A(t,x))\big)\partial_x\varphi \\ &- \mathrm{sgn}(\rho-\kappa)\partial_A f(\kappa, A(t,x))\partial_x(A(t,x))\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} |\rho_0(x)-\kappa|\varphi(0,x)\,\mathrm{d}x \geq 0 \end{aligned} \tag{1.1.4}$$

for all $\varphi \in \mathrm{C}^1_c([0,T)\times\mathbb{R};\mathbb{R}_+)$ and $\kappa \in \mathbb{R},$ where sgn is the sign function.

Now, based on the nature of the kernel function μ, we split (1.1.1) into two distinct models, which form our primary focus.

Model 1. In this model, the flux function in (1.1.1) takes the form $f(\rho, A) = g(\rho)v(A)$, where $v \in \mathrm{C}^2(I;\mathbb{R}_+)$ with $v' \leq 0$ and $g \in \mathrm{C}^1(I;\mathbb{R}_+)$ with $g' \geq 0$. Further, the kernel function μ is given by

$$\mu(x) = w_\eta(-x),$$

where $\eta > 0$ is fixed, $w_\eta \in C^1([0,\eta];\mathbb{R}_+)$, $\mathrm{supp}(w_\eta) \subseteq [0,\eta]$, $w'_\eta \leq 0$, and $\int_0^\eta w_\eta(x)\,\mathrm{d}x = 1$. This gives rise to a convolution term of the form

$$A(t,x) = \int_x^{x+\eta} w_\eta(y-x)\rho(t,y)\,\mathrm{d}y, \tag{1.1.5}$$

which represents the downstream weighted average of ρ in the interval $[x, x+\eta]$.

This model is specifically formulated using a piecewise smooth, non-increasing kernel and a downstream convolution to describe traffic flow, incorporating evaluation of downstream density by drivers through the nonlocal term (1.1.5). This problem is originally proposed in [34] and have since been extensively studied (see [54, 89, 55, 88, 47]). More sophisticated variants of these equations have also been developed to model various physical settings, such as one-to-one junctions [52], on-off ramps [51], multilane traffic [87], networks [86], and flux-discontinuous models [50].

Model 2. This model arises from (1.1.1) by specifying the kernel as $\mu \in (\mathrm{C}^2_c \cap \mathrm{W}^{2,\infty})(\mathbb{R};\mathbb{R})$. Depending on the support of μ, this leads to possibly different orientations of the convolution integral. For a fixed $\eta > 0$, typical cases include:

- *Upstream kernel:* If $\mathrm{supp}(\mu) \subseteq [0,\eta]$, then
$$A(t,x) = \int_{x-\eta}^{x} \mu(x-y)\rho(t,y)\,\mathrm{d}y.$$
- *Centered kernel:* If $\mathrm{supp}(\mu) \subseteq [-\eta/2, \eta/2]$, then
$$A(t,x) = \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} \mu(x-y)\rho(t,y)\,\mathrm{d}y.$$
- *Downstream kernel:* If $\mathrm{supp}(\mu) \subseteq [-\eta, 0]$, then
$$A(t,x) = \int_{x}^{x+\eta} \mu(x-y)\rho(t,y)\,\mathrm{d}y.$$

The flexibility in the convolution type makes it suitable for a diverse range of applications, including sedimentation (centered convolution) [33] and traffic flow (downstream convolution) [19, 15]. The well-posedness and numerical approximation of such models have been studied in [33, 19, 14]. See also [15, 16, 17] for further theoretical and numerical developments on models of this kind.

1.1.1.1 Numerical approximation

A central challenge in designing numerical schemes for non-local conservation laws lies in the careful discretization of convolution integrals (1.1.2). Several first-order finite volume methods have been proposed addressing this issue. In contrast to local conservation laws, even first-order numerical schemes for non-local equations often lack monotonicity, a property that plays a crucial role in establishing stability and convergence analysis for the local case. The lack of monotonicity introduces additional difficulties, which have been addressed in the literature through alternative techniques to ensure the stability and convergence of the schemes. This has led to a well established theory for first-order schemes, see [33, 19, 14, 34, 89, 90].

Now, we give a brief sketch of certain available first-order finite volume approximations of (1.1.1), of the form:

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right), \tag{1.1.6}$$

where the numerical flux $F^n_{j+\frac{1}{2}}$ at the interface $x_{j+\frac{1}{2}}$ is chosen by carefully approximating the non-local term (1.1.2). Some commonly used examples of numerical fluxes for both models are given below.

Model 1

Lax-Friedrichs-type numerical flux (see [34, 53]):

$$F^n_{j+\frac{1}{2}}(\rho_j^n, \rho_{j+1}^n) := \frac{1}{2}\left(g(\rho_j^n)v(A_j^n) + g(\rho_{j+1}^n)v(A_{j+1}^n)\right) - \frac{\alpha}{2}(\rho_{j+1}^n - \rho_j^n), \tag{1.1.7}$$

where $A_j^n \approx \int_{x_j}^{x_j+\eta} \rho(t^n, y) w_\eta(y - x_j)\,\mathrm{d}y$ and $\alpha \geq 1$.

Godunov-type numerical flux (see [89]):

$$\begin{aligned} F^n_{j+\frac{1}{2}}(\rho_j^n, \rho_{j+1}^n) &:= \begin{cases} \min_{\rho\in[\rho_j^n, \rho_{j+1}^n]} g(\rho)v(A^n_{j+\frac{1}{2}}) & \text{if } \rho_j^n \leq \rho_{j+1}^n, \\ \max_{\rho\in[\rho_{j+1}^n, \rho_j^n]} g(\rho)v(A^n_{j+\frac{1}{2}}) & \text{if } \rho_j^n \geq \rho_{j+1}^n. \end{cases} \\ &= g(\rho_j^n)v(A^n_{j+\frac{1}{2}}), \end{aligned} \tag{1.1.8}$$

where $A^n_{j+\frac{1}{2}} \approx \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}+\eta} \rho(t^n, y) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y$.

Model 2

Lax-Friedrichs-type numerical flux (see [19, 14]):

$$F^n_{j+\frac{1}{2}}(\rho_j^n, \rho_{j+1}^n) := \frac{1}{2}\left(f(\rho_j^n, A^n_{j+\frac{1}{2}}) + f(\rho_{j+1}^n, A^n_{j+\frac{1}{2}})\right) - \frac{\alpha}{2\lambda}(\rho_{j+1}^n - \rho_j^n), \tag{1.1.9}$$

where $A^n_{j+\frac{1}{2}} \approx \int_{-\infty}^{\infty} \mu(x_{j+\frac{1}{2}} - y)\rho(t, y)\,\mathrm{d}y$ and $\alpha \in (0, 1/3)$. For the general problem described by Model 2, it is not straightforward to propose a Godunov scheme. However, for fluxes of the specific form $f(\rho, A) = h(\rho)v(A)$, a *Godunov-type* numerical flux is defined in [15] as

$$F^n_{j+\frac{1}{2}}(\rho_j^n, \rho_{j+1}^n) := v(A^n_{j+\frac{1}{2}})F_G(\rho_j^n, \rho_{j+1}^n), \tag{1.1.10}$$

where F_G is the Godunov numerical flux corresponding to a local conservation law with h as the flux function.

1.1.1.2 Well-posedness

For both models discussed above, uniqueness of solution is established by proving a result on the continuous dependence on the initial datum as given below (see [34, 53] for more details):

Theorem 1.1.3. *Let $\hat{\rho}$ and $\tilde{\rho}$ be entropy solutions of Model 1/ Model 2 corresponding to the initial datums $\hat{\rho}_0$ and $\tilde{\rho}_0$, respectively. Then for $T > 0$, there exists a constant $C_T > 0$ such that*

$$\|\hat{\rho}(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq C_T \|\hat{\rho}_0 - \tilde{\rho}_0\|_{\mathrm{L}^1(\mathbb{R})} \quad \forall t \in [0,T]. \tag{1.1.11}$$

In particular, the entropy solution to Model 1/ Model 2 is unique.

Remark 1.1.4. For the specific case of Model 1 with $g(\rho) = \rho$, it was shown in [119] that the weak solutions (1.1.1) are unique, thus making the entropy criterion (1.1.4) redundant. However, in the general case, the entropy condition is required to ensure the uniqueness of solutions to (1.1.1).

To show the existence of an entropy solution, the typical strategy (see [34, 33]) is to show that under suitable hypothesis on the flux function f in (1.1.1) (under the assumptions of Model 1/Model 2) and an appropriate CFL restriction on the time step, for any fixed time T, the approximate solutions $\{\rho_\Delta\}_{\Delta>0}$ generated by a first-order scheme (for example Lax-Friedrichs or Godunov) satisfy the following results.

Theorem 1.1.5. *(L^∞ estimate, [33, 19]) If the initial datum $\rho_0 \in \mathrm{L}^\infty(\mathbb{R};\mathbb{R}_+)$, then there exists a constant $C_\infty > 0$ such that the approximate solutions satisfy the L^∞ estimate*

$$\|\rho_\Delta(t,\cdot)\| \leq C_\infty, \tag{1.1.12}$$

for all $t \in [0,T]$.

Theorem 1.1.6. *(BV estimate, [33, 19]) If the initial datum $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$, then the approximate solutions satisfy the bounded variation estimate*

$$\mathrm{TV}(\rho_\Delta(t,\cdot)) \leq C_{\mathrm{BV}}, \tag{1.1.13}$$

for all $t \in [0,T]$ and for some constants $C_{\mathrm{BV}} > 0$.

Theorem 1.1.7. *(L^1 Lipschitz continuity in time, [33, 19]) If the initial datum $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$, then there exists a constant $C_{Lip} > 0$ such that the approximate solutions satisfy*

$$\|\rho_\Delta(t_1,\cdot) - \rho_\Delta(t_2,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq C_{Lip}(|t_1 - t_2| + \Delta t), \tag{1.1.14}$$

for any $t_1, t_2 \in [0,T]$.

Theorem 1.1.8. *(Discrete entropy condition, [33, 19, 14]) If the initial datum* $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$, *then the approximate solutions satisfy a discrete entropy condition:*

$$\begin{aligned} |\rho_j^{n+1}-\kappa| - |\rho_j^n-\kappa| + \lambda\left(\Phi^n_{j+\frac{1}{2}}(\rho_j^n,\rho_{j+1}^n) - \Phi^n_{j-\frac{1}{2}}(\rho_{j-1}^n,\rho_j^n)\right) \\ + \frac{1}{2}\lambda\,\mathrm{sgn}(\rho_j^{n+1}-\kappa) f(\kappa, A^n_{j+\frac{1}{2}}) - f(\kappa, A^n_{j-\frac{1}{2}}) \le 0, \quad \text{for } \kappa \in \mathbb{R}, \end{aligned} \tag{1.1.15}$$

where $\Phi^{n,\kappa}_{j+\frac{1}{2}}(u,v) := F^n_{j+\frac{1}{2}}(u\vee\kappa, v\vee\kappa) - F^n_{j+\frac{1}{2}}(u\wedge\kappa, v\wedge\kappa)$, *and we define* $u \vee v := \max\{u,v\}$, $u\wedge v := \min\{u,v\}$.

In light of Theorems 1.1.5, 1.1.6 and 1.1.7, an application of the Kolmogorov's compactness theorem (Theorem A.8, [108]) guarantees the existence of a subsequence of approximate solutions converges to a function ρ in $C([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$. We note that instead of proving Theorem 1.1.7, an alternative approach (see [34, 53]) is to establish a space-time total variation estimate of the form:

Theorem 1.1.9. *(Space-time* BV *estimate [34, 53]) If the initial datum* $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$, *then the approximate solutions satisfy*

$$\mathrm{TV}(\rho_\Delta(\cdot,\cdot)) \le K_{\mathrm{BV}}, \tag{1.1.16}$$

for some constant $K_{\mathrm{BV}} > 0$.

Theorems 1.1.5 and 1.1.9 together with an application of the Helly's theorem (see Lemma 21.4, Chapter 5, [84]) ensures the existence of a convergent sequence of approximate solutions. Once a convergent subsequence of approximate solutions is obtained, a Lax-Wendroff-type ([127]) argument adapted to the non-local scenario (see for example [33, 34]) is used to show that the limit ρ is a weak solution of the problem (1.1.1). Further, using the discrete entropy condition (1.1.15) satisfied by the scheme under consideration (see Theorem 1.1.8), it can be shown that the limit is an entropy solution (1.1.4), leading to the convergence theorem:

Theorem 1.1.10. *Let* $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$. *Under a suitable CFL condition, the approximate solutions* ρ_Δ *converge to the unique entropy solution of the problem* (1.1.1).

1.1.2 Multi-dimensional system case

We consider the multidimensional extension of Model 2 discussed in Section 1.1.1 which is relevant in crowd dynamics and related phenomena [69, 67, 14]. For the sake of simplicity, we restrict our attention to two dimensions, and consider the Cauchy problem for the system:

$$\begin{aligned} \partial_t\boldsymbol{\rho} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{F}(t,\boldsymbol{x},\boldsymbol{\rho},\boldsymbol{\eta}_1*\boldsymbol{\rho},\cdots,\boldsymbol{\eta}_n*\boldsymbol{\rho}) = 0, \\ \boldsymbol{\rho}(\boldsymbol{x},0) = \boldsymbol{\rho}_0(\boldsymbol{x}). \end{aligned} \tag{1.1.17}$$

where $\boldsymbol{x} := (x, y)$, the unknown is

$$\boldsymbol{\rho} := \left(\rho^1, \rho^2, \cdots, \rho^N\right),$$

and the flux function takes the form

$$\boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\eta} * \boldsymbol{\rho}, \boldsymbol{\nu} * \boldsymbol{\rho}) := \begin{pmatrix} f^1(t, x, y, \rho^1, \boldsymbol{\eta} * \boldsymbol{\rho}) & g^1(t, x, y, \rho^1, \boldsymbol{\nu} * \boldsymbol{\rho}) \\ \vdots & \vdots \\ f^N(t, x, y, \rho^N, \boldsymbol{\eta} * \boldsymbol{\rho}) & g^N(t, x, y, \rho^N, \boldsymbol{\nu} * \boldsymbol{\rho}) \end{pmatrix}^T.$$

The convolution kernel functions corresponding to the x-and y-direction are then given by the matrices

$$\boldsymbol{\eta} := \boldsymbol{\eta}_1 = \begin{pmatrix} \eta^{1,1} & \cdots & \eta^{1,N} \\ \vdots & \ddots & \vdots \\ \eta^{m,1} & \cdots & \eta^{m,N} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu} := \boldsymbol{\eta}_2 = \begin{pmatrix} \nu^{1,1} & \cdots & \nu^{1,N} \\ \vdots & \ddots & \vdots \\ \nu^{m,1} & \cdots & \nu^{m,N} \end{pmatrix},$$

respectively, and the convolution terms are defined as

$$\begin{aligned} (\boldsymbol{\eta} * \boldsymbol{\rho})_q (t, x, y) &:= \sum_{k=1}^{N} \int\int_{\mathbb{R}^2} \eta^{q,k}(x - x', y - y')\rho^k(t, x', y') \,\mathrm{d}x' \,\mathrm{d}y', \\ (\boldsymbol{\nu} * \boldsymbol{\rho})_q (t, x, y) &:= \sum_{k=1}^{N} \int\int_{\mathbb{R}^2} \nu^{q,k}(x - x', y - y')\rho^k(t, x', y') \,\mathrm{d}x' \,\mathrm{d}y', \end{aligned}$$

for $q \in \{1, 2, \ldots, m\}$, where $\eta^{l,k}, \nu^{l,k} : \mathbb{R}^n \to \mathbb{R}$, for $l \in \{1, 2, \cdots m\}$ and $k \in \{1, 2, \cdots, N\}$. In [14], a dimensionally split first-order scheme was proposed to approximate (1.1.17), making use of a Lax-Friedrichs type numerical flux. Further, under necessary assumptions on the flux function and a suitable CFL restriction on the time step, the approximate solutions $\{\boldsymbol{\rho}_\Delta\}_{\Delta>0}$ were shown to satisfy the following results:

Theorem 1.1.11. *(L^∞ estimate, [14]) If the initial datum $\boldsymbol{\rho}_0 \in \mathrm{L}^\infty(\mathbb{R}^2; \mathbb{R}^N_+)$, then there exists a constant $C_\infty > 0$ such that the approximate solutions satisfy the L^∞-estimate*

$$\|\boldsymbol{\rho}_\Delta(t, \cdot, \cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^2;\mathbb{R}^N)} \leq C_\infty, \tag{1.1.18}$$

for all $t \in [0, T]$.

Theorem 1.1.12. *(BV estimate, [14]) If the initial datum $\boldsymbol{\rho}_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R}^2; \mathbb{R}^N_+)$, then for each k, the approximate solutions satisfy*

$$\mathrm{TV}(\rho^k_\Delta(t, \cdot, \cdot)) \leq C_{\mathrm{BV}}, \tag{1.1.19}$$

for all $t \in [0, T]$ and for some constants $C_{\mathrm{BV}} > 0$.

Theorem 1.1.13. *(L^1 Lipschitz continuity in time, [14]) If the initial datum $\boldsymbol{\rho}_0 \in L^\infty \cap BV(\mathbb{R}^2;\mathbb{R}_+^N)$, then there exists a constant $C_{Lip} > 0$ such that the approximate solutions satisfy*

$$\|\rho_\Delta^k(t^{n+1},\cdot,\cdot) - \rho_\Delta^k(t^n,\cdot,\cdot)\|_{L^1(\mathbb{R}^2)} \leq C_{Lip}\Delta t, \tag{1.1.20}$$

for any n such that $(n+1)\Delta t \leq T$.

These results, along with a discrete entropy estimate of the form (1.1.15), were established in [14]. By applying Kolmogorov's compactness theorem, the existence of a weak solution was obtained, as stated below.

Theorem 1.1.14. *([14]) Let the initial datum $\boldsymbol{\rho}_0 \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^2;\mathbb{R}_+^N)$. Under suitable assumptions on the flux function and a CFL condition on the time step, the approximate solutions converge, up to a subsequence, to a solution $\boldsymbol{\rho} \in C\left(\mathbb{R}_+; L^1(\mathbb{R}^2;\mathbb{R}_+^N)\right)$ of* (1.1.17), (4.1.2).

Remark 1.1.15. Although a discrete entropy inequality similar to (1.1.15) is obtained in [14] for the proposed scheme, in the absence of a standard Riemann semigroup (see Chapter 9, [36]), this does not single out a unique solution.

In computational fluid dynamics, first-order numerical schemes are generally considered robust and reliable, and they aid in establishing well-posedness of problems. However, second- or high-order methods offer the advantage of considerably more accurate solutions with the same computing cost, particularly for two- or three-dimensional problems. As a result, there has been a surge of research activities aimed at improving these high-order methods. In the context of non-local conservation laws, [57, 97] treat second-order schemes, while [48, 88] proposes and compares high-order discontinuous Galerkin methods and central WENO schemes. Despite this, a major gap remains: there are currently no rigorous convergence results available for second- or higher-order schemes in the non-local setting. This leads to the following crucial questions:

Q1. *Can we construct second-order accurate schemes that are rigorously proven to converge to the entropy weak solution for one-dimensional non-local conservation laws of the forms Model 1 and Model 2?*

Q2. *Is it possible to design second-order schemes with desirable theoretical properties for multi-dimensional systems of non-local conservation laws of the form* (1.1.17)*?*

As a part of this thesis, we aim to address these questions by developing second-order numerical schemes for non-local conservation laws, with a particular focus on establishing rigorous analytical results in both the one-dimensional and multi-dimensional problem cases.

1.2 Conservation laws with discontinuous flux

Conservation laws with discontinuous coefficients in the flux functions have been studied extensively in the past few decades. A general version of the Cauchy problem for a scalar conservation laws with discontinuous flux is given by

$$\begin{aligned} \partial_t \rho + \partial_x f(k(x), \rho) = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ \rho(x,0) = \rho_0(x), \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{1.2.1}$$

where t and x are the time and space variables, respectively and $u = u(x,t)$ is the unknown quantity. The coefficient $k(x)$ in the flux function f is allowed to be a discontinuous function of the spatial variable x.

Example 1. A classic example is the *two-flux* case, which emerges by specifying the flux f in (5.0.1) as follows:

$$f(H(x), \rho(x,t)) = H(x) f_r(\rho) + (1 - H(x)) f_l(\rho) = \begin{cases} f_l(\rho) & \text{for } x < 0, \\ f_r(\rho) & \text{for } x \geq 0, \end{cases} \tag{1.2.2}$$

where H is the Heaviside function.

Example 2. Another common example involves the flux function having a multiplicative dependency on a discontinuous coefficient, of the form:

$$f(k(x), \rho) = k(x) g(\rho), \tag{1.2.3}$$

for some function g.

One of the earliest instances where discontinuous flux was studied is in the context of traffic flow on roads with varying surface conditions [136]. Assuming that the different segments of the road admit different permissible velocities, this model proposed a prototypical example of a conservation law with discontinuous flux. The framework of discontinuous flux was later extended to the case of two-phase flow in porous media arising in oil reservoirs [94], sedimentation models [76], and so on (see [111, 45]).

These equations have been the focus of extensive theoretical and numerical studies over the past few decades, for a detailed review see [41, 134] and references therein. For flux functions with smooth spatial dependence, Kružkov [124] showed the existence and uniqueness of an entropy solution. However, standard theoretical tools and numerical methods are not applicable when the flux is discontinuous, posing key difficulties in the well-posedness and analysis of numerical solutions. This situation necessitates the development of novel theoretical approaches and specialized numerical methods. This topic has been extensively explored in the literature, where well-posedness is typically established through suitable numerical approximations; see, for example, [7, 37, 114, 116, 123]. For a comprehensive framework on the well-posedness of (5.0.1), we refer to [21]. Conservations laws with discontinuous flux coefficients have also been investigated from various other perspectives. Recent contributions include regularity results established in [91], error estimates for monotone flux functions in [27], flux stability under similar assumptions in [148], compactness estimates in [118], viscosity approximation in [113], and BV regularity of adapted entropy solutions in [93], among others.

Solutions to (1.2.1) are generally interpreted in the following weak sense.

Definition 1.2.1. (Weak solution) A function $\rho \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be a weak solution of (5.0.1) if it satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} (\rho \partial_t \phi + f(k(x), \rho) \partial_x \phi) \, \mathrm{d}t \, \mathrm{d}x + \int_{\mathbb{R}} \rho_0 \phi(x, 0) \, \mathrm{d}x = 0, \tag{1.2.4}$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Let the function k be piecewise C^1, with a finite set of discontinuities denoted by $D = \{x_1, x_2, \ldots, x_M\}$. If the weak solution admits the trace values $\rho(t, x_i^\pm)$ for almost every $t > 0$, then it is straightforward that the interface Rankine-Hugoniot relation

$$f(k(x_i^-), \rho(t, x_i^-)) = f(k(x_i^+), \rho(t, x_i^+)) \quad \text{for}\, a.e.\, t > 0,$$

holds for $i \in \{1, 2, \ldots, M\}$. A fundamental aspect in the context of discontinuous flux is the uniqueness of solutions to these problems, where an entropy condition (or admissibility condition) plays a crucial role in ensuring uniqueness by identifying physically relevant solutions. Several attempts have been made in the literature [94, 75, 111, 114, 7] to distill the physically relevant solution. A common feature of all these entropy frameworks is that away from the discontinuities of k, the solution satisfies a Kružkov-type interior entropy inequality.

Definition 1.2.2. (Interior entropy inequality) A weak solution ρ is said to satisfy the interior entropy condition if for all constants $c \in \mathbb{R}$,

$$\begin{aligned} \int\int_{\mathbb{R} \times \mathbb{R}_+} & (|\rho - c| \partial_t \phi + \operatorname{sgn}(\rho - c)(f(k(x), \rho) - f(k(x), c)) \partial_x \phi) \, \mathrm{d}x \, \mathrm{d}t \\ & + \int_{\mathbb{R}} |\rho_0 - c| \phi(x, 0) \, \mathrm{d}x \geq 0, \end{aligned} \tag{1.2.5}$$

holds for all non-negative test functions $\phi \in \mathcal{D}((\mathbb{R} \setminus D) \times \mathbb{R}_+)$, where D is the set of discontinuities of k.

However, the interior entropy inequality (1.2.5) is not enough to guarantee a unique entropy solution. This is elaborated below using an example from [7] for the two-flux case (1.2.2) with the following assumptions (also see Figure 1.1):

(**H1**) The functions f_l and f_r defined on the interval $[s, S]$ such that $f_l(s) = f_r(s)$ and $f_l(S) = f_r(S)$.

(**H2**) The functions f_l and f_r intersect at a point $\alpha \in (s, S)$, with $f_r'(\alpha) > 0$ and $f_l'(\alpha) < 0$.

(**H3**) f_l and f_r have a single minimum, attained at θ_l and θ_r, respectively and $f_l(\theta_l) \geq f_r(\theta_r)$.

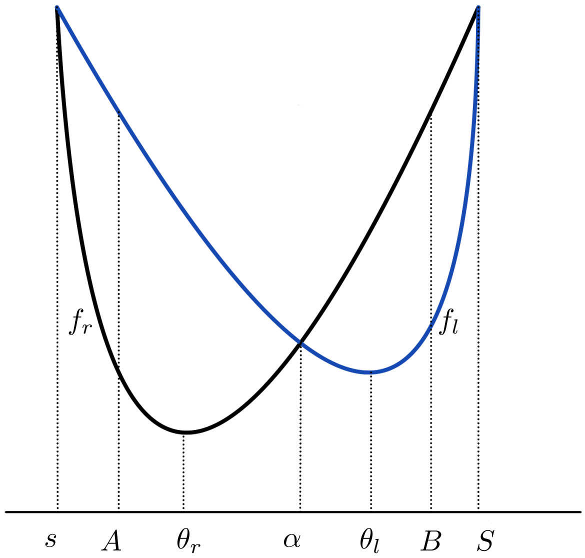


Figure 1.1: Left and right fluxes in (1.2.2) with the hypotheses (**H1**), (**H2**), (**H3**).

Now, consider the Cauchy problem for (1.2.2) with the initial datum

$$\rho_0(x) \equiv \alpha. \tag{1.2.6}$$

Let $A \in [s, \theta_l]$ and $B \in [\theta_r, S]$ be such that $f_l(A) = f_r(B)$. If $A \leq \alpha$, a solution to (1.2.2), (1.2.6) can be constructed as follows

$$\rho_{AB}(t, x) = \begin{cases} \alpha, & \text{if } x \leq \frac{f_l(A) - f_l(\alpha)}{A - \alpha} t, \\ A, & \text{if } \frac{f_l(A) - f_l(\alpha)}{A - \alpha} t < x < 0, \\ B, & \text{if } 0 < x < \frac{f_r(B) - f_r(\alpha)}{B - \alpha} t, \\ \alpha, & \text{if } \frac{f_r(B) - f_r(\alpha)}{B - \alpha} t \leq x. \end{cases} \tag{1.2.7}$$

On the other hand, if $A > \alpha$, then a solution can be constructed as follows

$$\rho_{AB}(t, x) = \begin{cases} \alpha, & \text{if } x \leq \frac{f_l(A) - f_l(\alpha)}{A - \alpha} t, \\ (f_l')^{-1}(\frac{x}{t}), & \text{if } \frac{f_l(A) - f_l(\alpha)}{A - \alpha} t < x < f_l'(A) t \\ A, & \text{if } f_l'(A) t < x < 0, \\ B, & \text{if } 0 < x < f_r'(B) t, \\ (f_r')^{-1}(\frac{x}{t}), & \text{if } f_r'(B) t < x < \frac{f_r(B) - f_r(\alpha)}{B - \alpha} t \\ \alpha, & \text{if } \frac{f_r(B) - f_r(\alpha)}{B - \alpha} t \leq x. \end{cases} \tag{1.2.8}$$

For each choice of the pair (A, B), we obtain a corresponding weak solution ρ_{AB} to the problem (1.2.2), (1.2.6) satisfying the interior entropy condition (1.2.5) and with trace values $\rho(t, 0^-) = A, \rho(t, 0^+) = B$. In other words, there are infinitely many weak solutions satisfying the interior entropy condition (1.2.5). To identify physically relevant solution among these, various selection criteria have been proposed in the literature. For instance, for the problem (1.2.2), (1.2.6), the frameworks in Gimse and Risebro [94], Diehl [75, 76], Karlsen, Risebro and Towers [114] choose the solution $\rho(t, x) \equiv \alpha$ which is corresponds to the choice $A = B = \alpha$. On the other hand, the entropy frameworks of Adimurthi and Gowda [5], Ostrov [140] and Kaasschieter [111] lead to the solution

$$\rho_{AB}(t,x) = \begin{cases} \alpha, & \text{if } x \leq \frac{f_l(A)-f_l(\alpha)}{A-\alpha}t, \\ (f_l')^{-1}(\frac{x}{t}), & \text{if } \frac{f_l(A)-f_l(\alpha)}{A-\alpha}t < x < f_l'(A)t \\ \bar{\theta}_l, & \text{if } 0 < x < f_r'(\bar{\theta}_l)t, \\ (f_r')^{-1}(\frac{x}{t}), & \text{if } f_r'(B)t < x < \frac{f_r(\bar{\theta}_l)-f_r(\alpha)}{\bar{\theta}_l-\alpha}t \\ \alpha, & \text{if } \frac{f_r(\bar{\theta}_l)-f_r(\alpha)}{\bar{\theta}_l-\alpha}t \leq x, \end{cases} \tag{1.2.9}$$

where $\bar{\theta}_l \in [\theta_f, S]$ such that $f_l(\theta_l) = f_r(\bar{\theta}_l)$, corresponding to the choice $(A, B) = (\theta_l, \bar{\theta}_l)$. In [7], for the two-flux case (1.2.2), all these solution notions were brought together under the unified framework known as (A, B) entropy solutions.

1.2.0.1 (A,B) entropy condition

First, we define an (A, B) entropy connection for the problem (1.2.2), with hypotheses (**H1**), (**H2**) and (**H3**).

Definition 1.2.3. ((A, B) interface connection) A pair (A, B), where $A \in [s, \theta_l]$, $B \in [\theta_r, S]$ is said to be an interface entropy connection if $f_l(A) = f_r(B)$.

Now, the interface entropy condition corresponding to the connection (A, B) is defined as follows.

Definition 1.2.4. (Interface entropy condition) A function ρ which admits traces $\rho(t, 0^\pm)$ for *a.e.* $t > 0$, is said to satisfy the interface entropy condition if

$$\begin{aligned} I_{AB}(t) := {} & \operatorname{sgn}(\rho(t, 0^-) - A)(f_l(\rho(t, 0^-)) - f_l(A)) \\ & - \operatorname{sgn}(\rho(t, 0^+) - B)(f_r(\rho(t, 0^+)) - f_r(B)) \geq 0 \end{aligned} \tag{1.2.10}$$

for a.e. $t > 0$.

Definition 1.2.5. ((A,B) entropy solution) A function $\rho \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be an entropy solution to (1.2.2) corresponding to the connection (A, B) if

- ρ is a weak solution as in (1.2.4),
- ρ satisfies the interior entropy condition (1.2.5), and
- ρ satisfies the interface entropy condition (1.2.10) corresponding to the connection (A, B).

Note that the existence of traces $\rho(t, 0^{\pm})$ is implicitly assumed in Definition (1.2.5) to satisfy the interface entropy condition. In [7], it was established that for the two-flux case (1.2.2) with (**H1**), (**H2**) and (**H3**), the (A, B) entropy solution corresponding to each connection (A, B) forms an L^1 contractive semigroup. This has led to the fact that there is no unique physically relevant solution for scalar conservation laws with discontinuous flux. Instead, the choice of the appropriate (A, B) connection is determined based on the physics of the problem. The (A, B) entropy framework was generalized to the case of more general flux functions and finitely many discontinuities in [10, 8, 9, 11].

1.2.0.2 Kružkov entropy condition

In this thesis, we focus on a Kružkov-type entropy formulation, mainly based on the framework of [116, 114]. For a strongly degenerate parabolic equation of the form:

$$\partial_t \rho + \partial_x f(k(x), \rho) = \partial^2_{xx} A(\rho), \tag{1.2.11}$$

where $A : \mathbb{R} \to \mathbb{R}$ is the diffusion function, a Kružkov-type entropy condition was derived in [114]. For (1.2.1), this entropy condition takes the following form.

Definition 1.2.6. (Kružkov entropy solution) A function $\rho \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is called an entropy solution of (5.0.1) if for all $c \in \mathbb{R}$,

$$\begin{aligned}
&\int\int_{\mathbb{R}\times\mathbb{R}_+} (|\rho - c|\partial_t \phi + \operatorname{sgn}(\rho - c)(f(k, \rho) - f(k, c))\partial_x \phi) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}} |\rho_0 - c|\phi(x, 0)\,\mathrm{d}x + \int\int_{(\mathbb{R}\setminus D)\times\mathbb{R}_+} |\partial_x f(k(x), c)|\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{m=1}^{M} \int_0^\infty |f(k_m^+, c) - f(k_m^-, c)|\phi(x_m, t)\,\mathrm{d}t \geq 0,
\end{aligned} \tag{1.2.12}$$

for all non-negative test functions $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$, where $D = \{x_1, x_2, \ldots, x_M\}$ is the set of discontinuities of k and $k_m^{\mp}$ denote the left and right traces of k at x_m, respectively.

Away from the discontinuities of k, the condition (1.2.12) ensures that the interior entropy condition (1.2.5) holds. The uniqueness of the entropy solution, as defined in (1.2.12), was established in [114] under the assumption of a *crossing condition* on the flux function f.

Definition 1.2.7. (Crossing condition) For any jump in the coefficient k with the corresponding left and right limits k_m^- and k_m^+ respectively,

$$f(k_m^+, u_1) - f(k_m^-, u_1) < 0 < f(k_m^+, u_2) - f(k_m^-, u_2) \implies u_1 < u_2, \tag{1.2.13}$$

for any states u_1, u_2.

In the two-flux case (1.2.2), under assumptions (**H1**), (**H2**), and (**H3**), the Kružkov-type entropy formulation (1.2.12) corresponds to the (A, B) entropy solution (as defined in Definition (1.2.5)) for a specific choice of the pair (A, B). For instance, the constant solution $\rho(t, x) \equiv \alpha$ is a Kružkov entropy solution of the Cauchy problem (1.2.2), (1.2.6) in the sense of (1.2.12), and it coincides with the (A, B) entropy solution corresponding to the choice $A = B = \alpha$. However, the (A, B) entropy framework introduced in [7] is restrictive in several important situations, as it relies on the assumption that the flux functions are unimodal (i.e., possessing a single extremum). In many relevant physical scenarios, including clarifier-thickener models [42, 43, 45], the flux function may have both a local maximum and a local minimum. Although the (A, B) entropy framework was extended in [8] to cover flux functions with multiple extrema, this generalization still requires that f_l' and f_r' have opposite signs at all points of intersection, an assumption that does not hold in general for clarifier-thickener models. In contrast, the Kružkov-type entropy formulation (1.2.12) remains applicable in such cases (see [77]). Moreover, this formulation can be interpreted as the unique vanishing viscosity solution, a notion that has been rigorously studied and validated in several works [77, 22, 25, 76, 114, 141, 149, 160] and remains relevant across various models that incorporate discontinuous flux.

In [77], for the two-flux case with general flux functions and multiple flux crossings, the entropy condition (1.2.12) was generalized through the so-called Γ-condition, which eliminates the need for the crossing condition. When the crossing condition is satisfied, the solutions chosen by the Γ-condition coincides with the Kružkov-type solution (1.2.12). Another approach towards the elimination of the crossing condition can be found in [46], which proposes an (A,B) connection adapted Kružkov entropy framework for the two-flux case with unimodal flux functions and a single flux crossing. Furthermore, in [117], a Kružkov-type adapted entropy condition was shown to yield uniqueness for more general flux functions, even in the absence of a crossing condition.

1.2.0.3 Numerical approximation

A wide variety of numerical techniques have been developed in the literature to approximate conservation laws with discontinuous flux. While the list is extensive, we mention a few approaches: Godunov-type schemes [7, 20, 92, 117], relaxation schemes [112], Enguist-Osher schemes [46, 160] and upstream mobility schemes [135]. Other approaches include DFLU flux [13], Roe-type schemes [169], and Monte-Carlo methods for random conservation laws with discontinuous coefficients [26]. Furthermore, general monotone (A, B) entropy stable schemes and a local Lax-Friedrichs scheme were analyzed in [4] and [154], respectively.

A crucial feature of conservation laws with discontinuous flux functions is that solutions may fail to have bounded total variation; see [3, 93]. Since total variation boundedness is a central tool in proving the convergence of numerical schemes, this presents a significant challenge in the analysis of numerical schemes. A widely adopted strategy to overcome this is the singular mapping technique, wherein the images of the approximate solutions under a suitable monotone map are shown to have diminishing total variation. This technique was introduced by Temple in [157] to prove the convergence of the Glimm scheme, specifically applied to a 2×2 resonant system of conservation laws for modeling oil displacement by water and polymer in reservoirs (see [6]). An alternative theoretical framework for convergence is the compensated compactness framework [79, 156], which was generalized to the case of discontinuous flux problems in [116]. This relies on a L^∞ estimate on the approximate solutions and the $\mathrm{W}^{-1,2}_{\mathrm{loc}}$ compactness of terms involving approximate solutions.

Specifically, for the general problem (1.2.1), a staggered Lax-Friedrichs-type scheme was proposed in [116]:

$$\rho^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}(\rho^n_j + \rho^n_{j+1}) - \lambda \left(f(k_{j+1}, \rho^n_{j+1}) - f(k_j, \rho^n_j) \right), \tag{1.2.14}$$

Under appropriate assumptions on the flux function (including the crossing condition (1.2.13)) and a suitable CFL condition on the time step, the scheme (1.2.14) was shown to satisfy the following results:

Theorem 1.2.8. *(Maximum principle, [116]) Let the initial datum $\rho_0 \in \mathrm{L}^\infty(\mathbb{R})$ with $a \leq \rho_0(x) \leq b$, for all $x \in \mathbb{R}$. The approximate solutions $\{\rho_\Delta\}_{\Delta>0}$ obtained from the scheme* (1.2.14) *satisfies the global maximum principle*

$$a \leq \rho_\Delta(x,t) \leq b, \tag{1.2.15}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Theorem 1.2.9 ($\mathrm{W}^{-1,2}_{\mathrm{loc}}$*compactness*, [116])**.** *Let $\rho_0 \in \mathrm{L}^\infty(\mathbb{R})$. For*

$$\begin{aligned} S_1(\rho) &:= \rho - c, \quad Q_1(k,\rho) := f(k,\rho) - f(k,c), \\ S_2(k,\rho) &:= f(k,\rho) - f(k,c) \quad \text{and} \quad Q_2(k,\rho) := \int_c^\rho (f_\rho(k,\xi))^2 \, \mathrm{d}\xi, \end{aligned} \tag{1.2.16}$$

the sequence of distributions

$$\{S_i(k(x),\rho_\Delta)_t + Q_i(k(x),\rho_\Delta)_x\}_{\Delta>0},$$

is contained in a compact subset of $\mathrm{W}^{-1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$, for $i = 1,2$ and for any $c \in \mathbb{R}$.

In [116], these results followed by the application of the compensated compactness theorem provides a convergent sequence of approximate solutions which converge to a weak solution. Further, a discrete cell entropy inequality is derived for the scheme (1.2.14):

$$\begin{aligned} |\rho^{n+1}_{j+\frac{1}{2}} - c| - \frac{1}{2}|\rho^n_{j+1} - c| - \frac{1}{2}|\rho^n_j - c| + \lambda \left(F(k_{j+1}, \rho^n_{j+1}, c) - F(k_j, \rho^n_j, c)\right) \\ - \lambda \operatorname{sign}(\rho^{n+1}_{j+\frac{1}{2}} - c)(f(k_{j+1}, c) - f(k_j, c)) \leq 0, \end{aligned} \tag{1.2.17}$$

for $F(k, \rho, c) := \operatorname{sgn}(\rho - c)(f(k, \rho) - f(k, c))$. Using (1.2.17), convergence of the approximate solutions computed from (1.2.14) to the entropy solution (1.2.12) is established, thus showing the existence of an entropy solution.

Despite the advancements in first-order numerical methods, rigorous analytical results for second- and higher-order methods remain limited. Some progress in this direction can be found in [12, 44, 153]. In particular, the convergence analysis of a class of second-order schemes to a weak solution was studied in [44]. This analysis was later extended in [12] to accommodate a broader class of numerical fluxes and to establish convergence to the (A, B)-entropy solution. Nonetheless, these studies rely on an additional non-local limiter algorithm to ensure that the scheme is FTVD (flux total variation diminishing). While effective, the limiter algorithm introduces additional computational tasks compared to conventional second-order schemes. This naturally leads to the question:

Q3. *Is it possible to design a relatively simple scheme for* (1.2.1)*, such as one based on MUSCL-type spatial reconstruction, and establish its convergence to the entropy solution?*

To the best of our knowledge, this remains an open question, as also noted in [44]. This constitutes another major problem addressed in this thesis.

1.3 Contributions of the thesis

Addressing the questions Q1, Q2, and Q3, this thesis proposes second-order numerical schemes for classes of conservation laws with non-local and discontinuous fluxes, introduced in Sections 1.1 and 1.2. A central focus is the derivation of rigorous analytical results that establish the reliability of these schemes. The main contributions are presented in detail in the following chapters, which are organized as follows.

Chapter 2: Second-order schemes for non-local traffic flow models. For non-local conservation laws of the form Model 1, we construct a fully discrete second-order scheme by combining a MUSCL-type spatial reconstruction with strong stability-preserving Runge–Kutta time integration. To establish convergence, we derive several key estimates for the scheme: a maximum principle, bounded variation bounds, and L^1-Lipschitz continuity in time. Applying these results within the framework of Kolmogorov's compactness theorem yield a convergent subsequence of approximate solutions. The limit is shown to be a weak solution using an explicit Lax–Wendroff-type argument adapted to the non-local setting. As deriving a suitable discrete entropy inequality for second-order schemes remains elusive, establishing entropy convergence is not straightforward. We overcome this difficulty by introducing a mesh-dependent slope limiter that essentially ensures convergence to the entropy solution through a sequence of carefully constructed theorems and lemmas. We also propose a MUSCL–Hancock (MH)-type second-order scheme that requires only a single intermediate reconstruction stage. However, a rigorous convergence analysis for this scheme remains out of reach. Several numerical experiments are presented to verify the theoretical results and to demonstrate the superior accuracy of the second-order schemes over a first-order method. Among the second-order schemes, the MH-type scheme demonstrates slightly better resolution and greater computational efficiency, as expected from the inherent structure of the MH scheme.

Chapter 3: MUSCL–Hancock scheme for non-local conservation laws. For the discretization of non-local conservation laws of the form Model 2, we develop a single-stage MUSCL–Hancock-type second-order scheme. The MUSCL–Hancock (MH) scheme is well known for its computational efficiency compared to other standard two-stage second-order methods, such as the MUSCL–Runge–Kutta scheme. In the context of non-local conservation laws, the main difficulty in designing a MH scheme arises from the discretization of the convolution term. We address this by employing a piecewise linear reconstruction of the discrete convolution, which is carefully constructed to preserve second-order accuracy and to support a rigorous convergence analysis. To establish the convergence of the proposed scheme, we derive several key estimates, including an L^∞ bound, bounded variation estimates, and L^1-Lipschitz continuity in time. These estimates, along with the Kolmogorov's compactness theorem, allow us to obtain the convergence of a subsequence of approximate solutions to a weak solution. It is worth noting that, as in the previous case, the lack of a suitable discrete entropy inequality for second-order schemes makes proving convergence to the entropy solution particularly challenging. To address this issue, we introduce a mesh-dependent modification of the minmod slope limiter and prove convergence to the entropy solution via a sequence of suitably formulated results. Additionally, we present numerical results to validate the theoretical findings. These results illustrate that the proposed scheme significantly improves accuracy over a first-order counterpart while also being computationally more efficient than standard two-stage second-order schemes.

Chapter 4: Second-order scheme for multidimensional system of non-local conservation laws. In this study, we consider the general class (1.1.17) of non-local systems of conservation laws in multiple space dimensions and propose a fully discrete second-order scheme by combining a MUSCL-type spatial reconstruction and a Runge-Kutta time integration. By reformulating the scheme appropriately, we prove that the proposed scheme preserves the positivity of all the unknowns, a critical property for ensuring the physical validity of quantities like density, which must remain non-negative. Additionally, we establish that the scheme is L^∞-stable. Numerical experiments are conducted for two cases: a scalar crowd dynamics model and a non-local Keyfitz-Kranzer system. The results illustrate the superior performance of the second-order method compared to that of a first-order implementation and confirm the theoretical analysis. The robustness of the scheme is further tested numerically in the 'singular limit problem', showing that as the non-local parameter tends to zero, the solutions converge to the local problem at a higher rate than with a first-order method.

Chapter 5: A MUSCL-type central scheme for conservation laws with discontinuous flux.

Focusing on the numerical discretization of of the general class (1.2) of scalar conservation laws with discontinuous flux, we propose a second-order MUSCL-type central scheme and present its convergence analysis. Since solutions to problems with discontinuous flux generally do not belong to the space of bounded variation (BV), the singular mapping technique is the commonly used approach to establish the convergence of numerical schemes. However, since this framework demands the monotonicity of the scheme, this is not applicable for MUSCL-type second-order schemes. To overcome this difficulty, we employ the theory of compensated compactness to show the convergence. A major component of our analysis involves deriving the maximum principle and showing the $\mathrm{W}^{-1,2}_{\mathrm{loc}}$ compactness of a sequence constructed from approximate solutions. The latter is achieved through the derivation of several essential estimates on the approximate solutions, including a cubic estimate. The compensated compactness theorem then guarantees the existence of a subsequence that converges to a weak solution. Further, as a key novelty of our approach, we show that the cubic estimate, which is significantly weaker than a BV estimate, is sufficient to develop an entropy convergence framework for a class of schemes in the predictor-corrector form. Within this framework, we establish that a mesh-dependent correction in the slope limiter ensures the convergence of the proposed scheme to the entropy solution.

Chapter 6: Conclusion.

We summarize the investigations carried out in this thesis and highlight the main contributions.

2

Second-order schemes for non-local traffic flow models

In this chapter, we focus on traffic flow models governed by non-local conservation laws (referred to as Model 1 in Chapter 1, Section 1.1), originally introduced in [34] and subsequently studied in [52, 55, 56, 89, 97]. These models capture driver behavior by incorporating the influence of surrounding vehicle density and have been the subject of extensive investigation in recent years. The primary objective of this chapter is to develop a second-order numerical scheme for the one-dimensional non-local traffic flow model considered in [53, 88, 89], and to establish its convergence through theoretical analysis. As discussed in Chapter 1, Section 1.1, second- or higher-order methods are essential for achieving improved accuracy and computational efficiency in the numerical approximation of solutions. In this direction, several works have been proposed for non-local traffic flow models: the authors in [57, 97] have developed second-order schemes, while others as in [48, 88] have proposed and compared high-order discontinuous Galerkin and central WENO methods. Despite these advancements, rigorous convergence results for second- or higher-order methods remain unavailable. This chapter aims to contribute in this direction by providing the convergence analysis of a second-order scheme.

To derive a second-order scheme, we employ a MUSCL-type spatial reconstruction [162] along with a strong stability preserving Runge-Kutta time stepping method [103, 104]. Such schemes are commonly used to discretize local conservation laws, for a detailed description we refer to [103, 151]. In [57], a second-order scheme of this type was presented for non-local multi-class traffic flow problems, showing its positivity preserving property analytically and through numerical examples. In this chapter, our focus is on the convergence analysis of this second-order scheme which we denote by RK-2. To obtain the convergence results, we also utilize a suitable numerical integration rule for approximating the convolution term. The convergence analysis involves two main stages. In the first stage, we aim to show that the proposed scheme converges to a weak solution. This is achieved by deriving a sequence of results that establish L^∞ estimates, Lipschitz continuity property in time and total variation (TV) bounds on the family of approximate solutions. We then use a version of the Kolmogorov's theorem to extract a subsequence that converges to a weak solution. Through the classical Lax-Wendroff type argument [128] we show that the limit of the convergent sequence is a weak solution of the given problem. Importantly, in a specific case (where $g(\rho) = \rho$, as we see later), weak solutions are already unique and an entropy condition is not necessary, see [119]. In the second stage, for the more general case, building on the ideas presented in [167], we employ a space-step dependent slope limiter (see [12, 165]) to establish the convergence to the entropy solution. Furthermore, we also consider a different type of non-local traffic-flow model, referred to as the downstream velocity model, proposed in [89]. We note that the convergence analysis presented in this chapter is applicable to this model as well.

In addition to this, we propose a MUSCL-Hancock type second-order scheme for the non-local problems of [53, 88, 89]. We denote this scheme by MH. The MUSCL-Hancock scheme, initially introduced in [163] and subsequently explored in [31, 164], is widely recognized for its simplicity and accurate shock capturing capabilities. In our work, we have tailored this approach to create a second-order scheme through appropriate numerical integration of the non-local flux term. Our analysis shows that the MH scheme provides a solution that is comparable to that of RK-2, while requiring only one stage of spatial reconstruction. This characteristic would save computational time, particularly when dealing with two-dimensional problems. However, the convergence analysis of this scheme applied to the traffic flow model described in Model 1 is deferred to future work.

The rest of the chapter is organized as follows: Section 2.1 describes the mean-downstream density traffic flow problem and discusses the notion of weak and entropy solutions of the underlying problem. Next, in Section 2.2, we present a second-order scheme that combines a MUSCL-type spatial reconstruction and Runge-Kutta time stepping to solve the underlying problem numerically. Further, we demonstrate that the approximate solutions obtained using this scheme possess desirable properties such as the maximum principle, a BV estimate, and L^1-Lipschitz continuity in time. In Section 2.3, we prove the existence of a subsequence which converges to a weak solution of the given problem. In Section 2.4, we establish the convergence of the scheme to the entropy solution. In Section 2.5, we propose a MUSCL-Hancock type second-order accurate scheme for the approximation of non-local traffic flow problems. Finally, in Section 2.6, we provide numerical experiments to illustrate the performance of the second-order schemes in comparison to a first-order scheme. We conclude with our final remarks in Section 2.7. Appendix A.1 contains the proof of Kolmogorov's theorem adapted to our context; Appendix A.2 provides a description of the mean downstream velocity model; and Appendix A.3 presents supporting results used in proving convergence to the entropy solution.

2.1 Mean downstream density traffic flow model

We consider the following initial value problem for the non-local conservation law, originally proposed in [34, 53]:

$$\begin{aligned} \partial_t \rho + \partial_x\big(g(\rho)v(\rho * w_\eta)\big) &= 0, \quad x \in \mathbb{R},\, t \in (0,T], \\ \rho(0,x) &= \rho_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{2.1.1}$$

which describes the evolution of the vehicle density $\rho(t,x)$. Here, g and the velocity v are given functions of the density such that the map $\rho \mapsto g(\rho)v(\rho)$ is the corresponding local conservation law flux. The function w_η is a convolution kernel with compact support in $[0,\eta]$ for some $\eta > 0$. The convolution term $\rho * w_\eta$ is defined as

$$\rho * w_\eta(t,x) \coloneqq \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y. \tag{2.1.2}$$

We denote $\mathbb{R}_+ \coloneqq [0,+\infty)$ and make the hypotheses on the functions v, g and w_η as follows: $v \in \mathrm{C}^2(I;\mathbb{R}_+)$ with $v' \leq 0, g \in \mathrm{C}^1(I;\mathbb{R}_+)$ with $g' \geq 0$, $w_\eta \in \mathrm{C}^1([0,\eta];\mathbb{R}_+)$ with $w_\eta' \leq 0, \int_0^\eta w_\eta(x)\,\mathrm{d}x = 1$, where $I = [0,\rho_{\max}] \subseteq \mathbb{R}_+, \rho_{\max} > 0$. Further, we assume $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$. Through the convolution of the density profile ρ with the kernel w_η, the non-local flux function denoted by $f(t,x,\rho) := g(\rho)v(\rho * w_\eta)$ takes into account the reaction of drivers to the neighbouring density of vehicles. In the case of traffic flow, the assessment of surrounding density generally happens only in the downstream direction by looking ahead, giving greater importance to closer vehicles. In this context, at a given time t, the velocity of cars at the point x has to be thought of as a function of not just the density $\rho(t,x)$ but of a weighted mean of the density in a right neighbourhood of x. This leads to the mean downstream density term $\rho * w_\eta$ as a convolution with a non-increasing kernel function w_η in the domain $[x, x+\eta]$ (see [34]). Through this mechanism, the non-local conservation law model turns to be suitable for describing traffic flow in a congested or heterogeneous road network. In general, the solutions of (2.1.1) need not be smooth, necessitating the definition of a weak solution given in the following lines.

Definition 2.1.1. (Weak solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R}; \mathbb{R})$, $T > 0$, is a weak solution of (2.1.1) if

$$\int_0^T \int_{-\infty}^{+\infty} \big(\rho \partial_t \varphi + g(\rho) v(\rho * w_\eta) \partial_x \varphi\big)(t,x) \,\mathrm{d}x \,\mathrm{d}t + \int_{-\infty}^{+\infty} \rho_0(x) \varphi(0,x) \,\mathrm{d}x = 0 \tag{2.1.3}$$

for all $\varphi \in \mathrm{C}_c^1([0,T) \times \mathbb{R}; \mathbb{R})$.

Further, we consider the following definition of entropy solution of (2.1.1) given in [53, 89].

Definition 2.1.2. (Entropy solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R}; \mathbb{R})$, $T > 0$, is an entropy weak solution of (2.1.1) if

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \Big(|\rho - \kappa| \partial_t \varphi + \operatorname{sgn}(\rho - \kappa) \big(g(\rho) - g(\kappa)\big) v(\rho * w_\eta) \partial_x \varphi \\
&- \operatorname{sgn}(\rho - \kappa) g(\kappa) v'(\rho * w_\eta) \partial_x (\rho * w_\eta) \varphi \Big)(t,x) \,\mathrm{d}x \,\mathrm{d}t + \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0,x) \,\mathrm{d}x \geq 0
\end{aligned} \tag{2.1.4}$$

for all $\varphi \in \mathrm{C}_c^1([0,T) \times \mathbb{R}; \mathbb{R}_+)$ and $\kappa \in I = [0, \rho_{\max}]$, where sgn is the sign function.

Note that, with this definition, uniqueness of entropy solutions of problem (2.1.1) follows from Theorem 2.1 of [53].

Remark 2.1.3. In a recent paper by Friedrich et al. [89], a new non-local conservation law model for describing traffic flow, known as the mean downstream velocity model, was introduced and studied. This model posits that drivers adjust their velocity based on the average velocity of vehicles in their vicinity. In our study, we will primarily examine the mean downstream density model, but it is worth noting that all of the results we present can be extended to the downstream velocity model as well. Additional information on this point can be found in Appendix A.2.

2.2 Second-order numerical scheme

To begin with, we discretize the spatial domain using a uniform mesh of size Δx, ensuring that the length of the convolution kernel's support $\eta = N\Delta x$ for some $N \in \mathbb{N}$. The spatial domain can then be represented as a union of cells, given by $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, where $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x$ for all $j \in \mathbb{Z}$. The time domain is discretized with grid points $t^n = n\Delta t$, where Δt is a time step which is chosen according to a CFL condition that will be specified later. Also, the ratio $\lambda = \frac{\Delta t}{\Delta x}$ is kept as a constant. Finally, we denote $w_\eta^k := w_\eta(k\Delta x)$ for $k = 0, \cdots, N$ and note the following properties

$$w_\eta^0 \geq w_\eta^k \text{ for all } k = 1, \cdots, N \tag{2.2.1}$$

and

$$\Delta x \sum_{k=0}^{N-1} w_\eta^k \leq w_\eta^0 N \Delta x = w_\eta^0 \eta. \tag{2.2.2}$$

Moving on to the numerical scheme, given the cell-average solution $\rho_j(t)$ at time t, we proceed with reconstructing a piecewise polynomial denoted as $\tilde{\rho}_j(t,x)$ which is given by

$$\tilde{\rho}_j(t,x) = \rho_j(t) + \frac{(x-x_j)}{\Delta x}\sigma_j(t) \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad j \in \mathbb{Z}. \tag{2.2.3}$$

Here,

$$\sigma_j(t) = \operatorname{minmod}\left((\rho_j(t)-\rho_{j-1}(t)),\ \frac{1}{2}(\rho_{j+1}(t)-\rho_{j-1}(t)), (\rho_{j+1}(t)-\rho_j(t))\right), \quad j \in \mathbb{Z} \tag{2.2.4}$$

represent the slopes obtained using the minmod limiter, where the minmod function is defined as

$$\operatorname{minmod}(a_1, \cdots, a_m) := \begin{cases} \operatorname{sgn}(a_1) \min\limits_{1\leq k\leq m}\{|a_k|\}, & \text{if } \operatorname{sgn}(a_1) = \cdots = \operatorname{sgn}(a_m), \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.5}$$

At each interface $x_{j+\frac{1}{2}}$, the terms $\rho_{j+\frac{1}{2},-}(t) := \rho_j(t) + \dfrac{\sigma_j(t)}{2}$ and $\rho_{j+\frac{1}{2},+}(t) := \rho_{j+1}(t) - \dfrac{\sigma_{j+1}(t)}{2}$ denote the left and right values of the reconstructed linear polynomial $\tilde{\rho}(t,x)$. With a finite volume integration, a spatially second-order semi-discrete scheme is obtained as

$$\frac{d\rho_j(t)}{dt} = -\frac{f_{j+\frac{1}{2}}(t) - f_{j-\frac{1}{2}}(t)}{\Delta x}, \quad \rho_j(0) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho_0(x)\,\mathrm{d}x \quad \text{for } j \in \mathbb{Z}, \tag{2.2.6}$$

where $f_{j+\frac{1}{2}}$ is the numerical flux. An immediate choice of the numerical flux is the Lax-Friedrich flux [34, 53]. However, we deal with a more accurate Godunov-type flux proposed in [89], given as

$$f_{j+\frac{1}{2}}(t) = g\left(\rho_{j+\frac{1}{2},-}(t)\right) V_{j+\frac{1}{2}}(t), \quad j \in \mathbb{Z},$$

where $V_{j+\frac{1}{2}}(t) := v\big(R_{j+\frac{1}{2}}(t)\big)$ and $R_{j+\frac{1}{2}}(t)$ denotes the approximation of the convolution term $R(t,x) := \rho * w_\eta(t,x)$ at the interface $x_{j+\frac{1}{2}}$. The terms $R_{j+\frac{1}{2}}(t)$ are computed using the trapezoidal rule as

$$R_{j+\frac{1}{2}}(t) = \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(\rho_{j+k+\frac{1}{2},+}(t) w_\eta^k + \rho_{j+k+\frac{3}{2},-}(t) w_\eta^{k+1}\right), \quad j \in \mathbb{Z}. \tag{2.2.7}$$

Finally, to obtain a second-order fully-discrete scheme, we evolve the semi-discrete formulation (2.2.6) in time using the strong stability preserving (SSP) Runge-Kutta method (as in [103, 151]). The resulting scheme is written as

$$\begin{aligned}\rho_j^{(1)} &= \rho_j^n - \lambda\left(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n - g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\right),\\ \rho_j^{(2)} &= \rho_j^{(1)} - \lambda\left(g(\rho_{j+\frac{1}{2},-}^{(1)})V_{j+\frac{1}{2}}^{(1)} - g(\rho_{j-\frac{1}{2},-}^{(1)})V_{j-\frac{1}{2}}^{(1)}\right),\\ \rho_j^{n+1} &= \frac{1}{2}\left(\rho_j^n + \rho_j^{(2)}\right), \quad j \in \mathbb{Z}.\end{aligned} \tag{2.2.8}$$

The second-order scheme (2.2.8) can also be written in the conservative form as

$$\rho_j^{n+1} = \rho_j^n - \lambda(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n), \tag{2.2.9}$$

where $F_{j+\frac{1}{2}}^n = \frac{1}{2}\left(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n + g(\rho_{j+\frac{1}{2},-}^{(1)})V_{j+\frac{1}{2}}^{(1)}\right), j \in \mathbb{Z}$. We denote the corresponding approximate solution as $\rho_{\Delta x}(t,x) := \rho_j^n$ for $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ and $\rho_{\Delta x}^n(x) := \rho_{\Delta x}(t^n, x)$. Also, define $\rho_{\Delta x}^{(l)}(t,x) := \rho_j^{(l)}$ for $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ and $l = 1, 2$. Here, $\rho_j^{(1)}$ and $\rho_j^{(2)}$ are taken to be the values computed from ρ_j^n for all $j \in \mathbb{Z}$, when $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$.

Remark 2.2.1. The linear reconstruction procedure (2.2.3) possesses the following properties which play an important role in the convergence analysis:
(i) The interface values have the property

$$\rho_{j+\frac{1}{2},-}(t), \rho_{j+\frac{1}{2},+}(t) \in \left[\min\{\rho_j(t), \rho_{j+1}(t)\}, \max\{\rho_j(t), \rho_{j+1}(t)\}\right], \quad j \in \mathbb{Z}. \tag{2.2.10}$$

(ii) The functions ρ and $\tilde{\rho}$ defined as $\rho(t,x) := \rho_j(t),\ \tilde{\rho}(t,x) := \tilde{\rho}_j(t,x),\ x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), j \in \mathbb{Z}$ satisfy the following equality (see Lemma 3.1, Chapter 4, [100]) on the total variation (TV):

$$\mathrm{TV}\big(\tilde{\rho}(t,\cdot)\big) = \mathrm{TV}\big(\rho(t,\cdot)\big) = \sum_{j\in\mathbb{Z}} |\rho_{j+1}(t) - \rho_j(t)|. \tag{2.2.11}$$

(iii) The slopes $\{\sigma_j(t)\}_{j\in\mathbb{Z}}$ satisfy

$$|\sigma_{j+1}(t) - \sigma_j(t)| \leq |\rho_{j+1}(t) - \rho_j(t)|, \quad \text{for all } j \in \mathbb{Z}. \tag{2.2.12}$$

Remark 2.2.2. To ensure non-negative velocity terms, we require that the convolution terms $R_{j+\frac{1}{2}}(t)$ in (2.2.7) fall within the range $[0, \rho_{\max}]$. In cases where the trapezoidal quadrature rule used to compute $R_{j+\frac{1}{2}}(t)$ in (2.2.7) is not exact for the given kernel function, i.e., $\frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w_\eta^k + w_\eta^{k+1}\right) \neq 1$, we adopt the same approach as in [88]. In this context, we define $Q_{\Delta x} := \frac{\Delta x}{2}\sum\limits_{k=0}^{N-1}\left(w_\eta^k + w_\eta^{k+1}\right)$ and replace w_η^k by

$$\tilde{w}_\eta^k = \frac{w_\eta^k}{Q_{\Delta x}}$$

so that

$$\frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(\tilde{w}_\eta^k+\tilde{w}_\eta^{k+1}\right)=1.$$

This allows us to write the term $R_{j+\frac{1}{2}}(t)$ in (2.2.7) as a convex combination of the density values $\rho_{j+k+\frac{1}{2},+}(t)$ and $\rho_{j+k+\frac{3}{2},-}(t)$, where $k=0,1,\ldots,N-1$. As a result, the convolution terms fall in the desired range provided the density values can be made to lie within the range $[0,\rho_{\max}]$, which we will see later. Consequently, the velocity terms remain non-negative. Moreover, replacing the terms w_η^k by $\tilde{w}_\eta^k$ does not affect the order of accuracy of the approximation as we have $Q_{\Delta x}\approx 1$. In addition, let us observe that, the modified terms $\tilde{w}_\eta^k$ also preserve the non-increasing property of w_η^k. With these observations, henceforth, in our convergence analysis of the RK-2 scheme (2.2.8), we replace w_η^k by $\tilde{w}_\eta^k$, while still denoting it as w_η^k.

2.2.1 Maximum principle

We establish that the approximate solution constructed using the scheme (2.2.8) satisfies the maximum principle. Initially, we examine the first-order forward Euler time stepping involved in the RK-2 scheme (2.2.8). Subsequently, we demonstrate the maximum principle for the RK-2 scheme (2.2.8). Now, we have the following lemma.

Lemma 2.2.3. *Let $\rho_j^n\in[\rho_m,\rho_M]\subseteq[0,\rho_{\max}]$ for all $j\in\mathbb{Z}$. Assume that the CFL condition*

$$\frac{\Delta t}{\Delta x}\leq\frac{1}{2(\|g\|\|v'\|\frac{\Delta x}{2}w_\eta^0+\|v\|\|g'\|)}\tag{2.2.13}$$

holds. Then the approximate solution obtained using the first-order Euler forward time step

$$\rho_j^{n+1}=\rho_j^n-\lambda\Big(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}}-g(\rho^n_{j-\frac{1}{2},-})V^n_{j-\frac{1}{2}}\Big)\tag{2.2.14}$$

satisfies $\rho_m\leq\rho_j^{n+1}\leq\rho_M$ for all $j\in\mathbb{Z}$.

Proof. Using the mean value theorem we write

$$V^n_{j-\frac{1}{2}}-V^n_{j+\frac{1}{2}}=v(R^n_{j-\frac{1}{2}})-v(R^n_{j+\frac{1}{2}})=-v'(\zeta_j)(R^n_{j+\frac{1}{2}}-R^n_{j-\frac{1}{2}}),\tag{2.2.15}$$

for some $\zeta_j\in\mathcal{I}(R^n_{j-\frac{1}{2}},R^n_{j+\frac{1}{2}})$. The difference of convolution terms reads as

$$\begin{aligned}&R^n_{j+\frac{1}{2}}-R^n_{j-\frac{1}{2}}\\&=\frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w_\eta^k(\rho^n_{j+k+\frac{1}{2},+}-\rho^n_{j+k-\frac{1}{2},+})+w_\eta^{k+1}(\rho^n_{j+k+\frac{3}{2},-}-\rho^n_{j+k+\frac{1}{2},-})\right).\end{aligned}\tag{2.2.16}$$

Using summation by parts yields

$$
\begin{aligned}
R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}} = \frac{\Delta x}{2}&\left(w_\eta^{N-1}\rho^n_{j+N-\frac{1}{2},+} - w_\eta^0 \rho^n_{j-\frac{1}{2},+} + \sum_{k=1}^{N-1} \rho^n_{j+k-\frac{1}{2},+}(w_\eta^{k-1} - w_\eta^k)\right) \\
+ \frac{\Delta x}{2}&\left(w_\eta^{N}\rho^n_{j+N+\frac{1}{2},-} - w_\eta^1 \rho^n_{j+\frac{1}{2},-} + \sum_{k=1}^{N-1} \rho^n_{j+k+\frac{1}{2},-}(w_\eta^{k} - w_\eta^{k+1})\right).
\end{aligned} \tag{2.2.17}
$$

Upon substituting the expression (2.2.17) into (2.2.15) and considering the assumptions that w_η is non-increasing, $w_\eta \geq 0$ and $v' \leq 0$ as well as the property (2.2.10) of the reconstructed values, it follows that

$$
\begin{aligned}
V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} &\leq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N-1}\rho_M - w_\eta^0\rho^n_{j-\frac{1}{2},+} + \rho_M \sum_{k=1}^{N-1}(w_\eta^{k-1} - w_\eta^k)\right) \\
&\quad - v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N}\rho_M - w_\eta^1\rho^n_{j+\frac{1}{2},-} + \rho_M \sum_{k=1}^{N-1}(w_\eta^{k} - w_\eta^{k+1})\right) \\
&\leq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N-1}\rho_M - w_\eta^0\rho^n_{j-\frac{1}{2},+} + \rho_M(w_\eta^0 - w_\eta^{N-1})\right) \\
&\quad - v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N}\rho_M - w_\eta^1\rho^n_{j+\frac{1}{2},-} + \rho_M(w_\eta^1 - w_\eta^{N})\right) \\
&= -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right).
\end{aligned} \tag{2.2.18}
$$

Subsequently, we get that $V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \leq \|v'\|\Delta x w_\eta^0 \rho_{\max}$. With a similar argument, we can show that

$$
V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \geq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^0(\rho_m - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_m - \rho^n_{j+\frac{1}{2},-})\right), \tag{2.2.19}
$$

which yields $V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \geq -\|v'\|\Delta x w_\eta^0 \rho_{\max}$. Consequently, we have

$$
|V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}}| \leq \|v'\|\Delta x w_\eta^0 \rho_{\max}. \tag{2.2.20}
$$

Multiplying the inequality in (2.2.18) with $g(\rho_M)$ and subtracting $g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}}$, we get

$$
\begin{aligned}
&V^n_{j-\frac{1}{2}} g(\rho_M) - V^n_{j+\frac{1}{2}} g(\rho^n_{j+\frac{1}{2},-}) \\
&\quad \leq \|g\|\|v'\|\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\
&\qquad + V^n_{j+\frac{1}{2}}\left(g(\rho_M) - g(\rho^n_{j+\frac{1}{2},-})\right) \\
&\quad \leq \|g\|\|v'\|\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\
&\qquad + \|v\|\|g'\|(\rho_M - \rho^n_{j+\frac{1}{2},-}).
\end{aligned} \tag{2.2.21}
$$

Given that the CFL condition (2.2.13) holds, the observation $\rho^n_j = \frac{1}{2}(\rho^n_{j-\frac{1}{2},+} + \rho^n_{j+\frac{1}{2},-})$ together with the monotonicity of g and the estimate (2.2.21) lead to the following estimate:

$$
\rho^{n+1}_j = \rho^n_j + \lambda\left(V^n_{j-\frac{1}{2}} g(\rho^n_{j-\frac{1}{2},-}) - V^n_{j+\frac{1}{2}} g(\rho^n_{j+\frac{1}{2},-})\right)
$$

$$
\begin{aligned}
&\le \rho_j^n + \lambda \left(V_{j-\frac12}^n g(\rho_M) - V_{j+\frac12}^n g(\rho_{j+\frac12,-}^n) \right) \\
&\le \frac{(\rho_{j-\frac12,+}^n + \rho_{j+\frac12,-}^n)}{2} + \lambda \|g\| \|v'\| \frac{\Delta x}{2} \Big(w_\eta^0 (\rho_M - \rho_{j-\frac12,+}^n) + w_\eta^1 (\rho_M - \rho_{j+\frac12,-}^n) \Big) \\
&\quad + \lambda \|v\| \|g'\| (\rho_M - \rho_{j+\frac12,-}^n) \\
&\le \left(\frac{\rho_{j-\frac12,+}^n}{2} + \lambda \|g\| \|v'\| \frac{\Delta x}{2} w_\eta^0 (\rho_M - \rho_{j-\frac12,+}^n) \right) \\
&\quad + \left(\frac{\rho_{j+\frac12,-}^n}{2} + \lambda \Big(\|g\| \|v'\| \frac{\Delta x}{2} w_\eta^1 + \|v\| \|g'\| \Big) (\rho_M - \rho_{j+\frac12,-}^n) \right) \\
&\le \left(\frac{\rho_{j-\frac12,+}^n}{2} + \frac{\rho_M - \rho_{j-\frac12,+}^n}{2} \right) + \left(\frac{\rho_{j+\frac12,-}^n}{2} + \frac{\rho_M - \rho_{j+\frac12,-}^n}{2} \right) = \rho_M. \qquad (2.2.22)
\end{aligned}
$$

Similarly, using (2.2.19) we can see that the following inequality holds

$$
\begin{aligned}
V_{j-\frac12}^n g(\rho_m) - V_{j+\frac12}^n g(\rho_{j+\frac12,-}^n) \ge \|g\| \|v'\| \frac{\Delta x}{2} \left(w_\eta^0 (\rho_m - \rho_{j-\frac12,+}^n) + w_\eta^1 (\rho_m - \rho_{j+\frac12,-}^n) \right) \\
+ \|v\| \|g'\| (\rho_m - \rho_{j+\frac12,-}^n).
\end{aligned}
$$

Using this inequality, in the same line of argument as in (2.2.22) we get the lower bound $\rho_j^{n+1} \ge \rho_m$, provided the CFL condition (2.2.13) holds. This concludes the proof of the lemma. ☐

Theorem 2.2.4. *Let $\rho_j^0 \in [\rho_m, \rho_M] \subset [0, \rho_{\max}]$ for all $j \in \mathbb{Z}$. Assume that the CFL condition* (2.2.13) *holds. Then for all $n \in \mathbb{N}$ the approximate solution obtained using the second-order scheme* (2.2.8) *satisfies*

$$
\rho_m \le \rho_j^n \le \rho_M \quad \textit{for all} \;\; j \in \mathbb{Z}. \qquad (2.2.23)
$$

Proof. Proof of this theorem uses the principle of mathematical induction. The base case $n = 0$ is trivially satisfied. For the inductive step, we assume that the inequality (2.2.23) holds for $n \in \mathbb{N}$, and show that it also holds for $n+1$. We use Lemma 2.2.3 to show that the first and second stages of the second-order scheme (2.2.8) also satisfy the maximum principle. First, we apply the Euler forward step to ρ_j^n to obtain $\rho_j^{(1)}$. By Lemma 2.2.3, we have $\rho_m \le \rho_j^{(1)} \le \rho_M$ for all $j \in \mathbb{Z}$. Next, we apply the Euler forward step to $\rho_j^{(1)}$ to obtain $\rho_j^{(2)}$. Again, by Lemma 2.2.3, we have $\rho_m^{(1)} \le \rho_j^{(2)} \le \rho_M^{(1)}$, where $\rho_M^{(1)} := \sup_{j \in \mathbb{Z}} \rho_j^{(1)}$ and $\rho_m^{(1)} := \inf_{j \in \mathbb{Z}} \rho_j^{(1)}$. Finally, as $\rho_j^{n+1} = \frac12 (\rho_j^n + \rho_j^{(2)})$, the result holds true for the case $n+1$. This completes the proof of the theorem. ☐

Remark 2.2.5. As a consequence of Theorem 2.2.4, it follows that the second-order scheme (2.2.8) is positivity preserving, in the sense that when the initial datum is positive then the approximate solution remains positive with evolution in time. Further, provided that the initial datum ρ_0 is positive, by using the conservative form (2.2.9) and the positivity property, it is immediate to see that the approximate solutions $\rho_{\Delta x}$ obtained using the scheme (2.2.8) satisfies the following

$$\|\rho_{\Delta x}(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} = \|\rho_{\Delta x}(0,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} = \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}, \quad \text{for all } t > 0.$$

2.2.2 Total variation estimate

We derive an estimate on the total variation of the approximate solutions obtained through the second-order scheme (2.2.8), when applied to the problem (2.1.1). This is done by initially examining the Euler forward step (2.2.14) and subsequently extending to the second-order scheme.

Lemma 2.2.6. *If* $\rho^n_{\Delta x} \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ *and the CFL condition* (2.2.13) *holds, then* $\rho^{n+1}_{\Delta x}$ *computed using the Euler forward step* (2.2.14) *has the space total variation estimate*

$$\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \le (1 + C\Delta t)\sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|, \tag{2.2.24}$$

where $C := w^0_\eta \rho_{\max}\|v'\|\|g'\| + 10\rho_{\max}(w^0_\eta)^2\|v''\|\|g\|\eta + 3w^0_\eta\|g\|\|v'\|$.

Proof. Let ρ^{n+1}_j be computed through (2.2.14). We proceed by subtracting and adding the term $-\lambda\big(V^n_{j+\frac{3}{2}} g(\rho^n_{j+\frac{1}{2},-}) + V^n_{j-\frac{1}{2}} g(\rho^n_{j+\frac{1}{2},-})\big)$ to the difference $\rho^{n+1}_{j+1} - \rho^{n+1}_j$. By using the observation $\rho^n_j = \frac{1}{2}(\rho^n_{j-\frac{1}{2},+} + \rho^n_{j+\frac{1}{2},-})$ and rearranging the terms, we obtain:

$$\begin{aligned}
&\rho^{n+1}_{j+1} - \rho^{n+1}_j \\
&\quad = (\rho^n_{j+1} - \rho^n_j) - \lambda\Big[V^n_{j+\frac{3}{2}}\big(g(\rho^n_{j+\frac{3}{2},-}) - g(\rho^n_{j+\frac{1}{2},-})\big) \\
&\qquad\quad - V^n_{j-\frac{1}{2}}\big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_{j-\frac{1}{2},-})\big) + g(\rho^n_{j+\frac{1}{2},-})(V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}})\Big] \\
&\quad = (\rho^n_{j+1} - \rho^n_j) - \lambda\Big[V^n_{j+\frac{3}{2}} g'(\theta_{j+1,-})\big(\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-}\big) \\
&\qquad\quad - V^n_{j-\frac{1}{2}} g'(\theta_{j,-})\big(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}\big) + g(\rho^n_{j+\frac{1}{2},-})\big(V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}}\big)\Big] \\
&\quad = \frac{1}{2}(\rho^n_{j+\frac{1}{2},+} - \rho^n_{j-\frac{1}{2},+}) + (\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-})\left(\frac{1}{2} - \lambda V^n_{j+\frac{3}{2}} g'(\theta_{j+1,-})\right) \\
&\qquad + \lambda V^n_{j-\frac{1}{2}} g'(\theta_{j,-})\big(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}\big) - \lambda g(\rho^n_{j+\frac{1}{2},-})\big(V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}}\big),
\end{aligned} \tag{2.2.25}$$

where $\theta_{j,-} \in \mathcal{I}(\rho^n_{j-\frac{1}{2},-}, \rho^n_{j+\frac{1}{2},-})$. We can write

$$\begin{aligned}
&V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}} \\
&\quad = (V^n_{j+\frac{3}{2}} - V^n_{j+\frac{1}{2}}) - (V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) \\
&\quad = v'(\zeta_{j+1})(R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) - v'(\zeta_j)(R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}) \\
&\quad = \big(v'(\zeta_{j+1}) - v'(\zeta_j)\big)(R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) + v'(\zeta_j)\big(R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}}\big) \\
&\quad = v''(\zeta_{j+\frac{1}{2}})(\zeta_{j+1} - \zeta_j)\big(R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}\big) + v'(\zeta_j)\big(R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}}\big),
\end{aligned} \tag{2.2.26}$$

where $\zeta_i \in \mathcal{I}(R^n_{i-\frac{1}{2}}, R^n_{i+\frac{1}{2}})$ and $\zeta_{i+\frac{1}{2}} \in \mathcal{I}(\zeta_i, \zeta_{i+1})$. By inserting the identity (2.2.26) into the expression (2.2.25), applying the CFL condition (2.2.13) and taking the modulus, it follows that

$$\begin{aligned} |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \le{}& \frac{1}{2}|\tilde{\Delta}^n_{j,+}| + |\tilde{\Delta}^n_{j+1,-}|\left(\frac{1}{2} - \lambda V^n_{j+\frac{3}{2}} g'(\zeta_{j+1,-})\right) + \lambda V^n_{j-\frac{1}{2}} g'(\zeta_{j,-})|\tilde{\Delta}^n_{j,-}| \\ &+ \lambda g(\rho^n_{j+\frac{1}{2},-})|v''(\zeta_{j+\frac{1}{2}})||\zeta_{j+1} - \zeta_j||R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}| \\ &+ \lambda g(\rho^n_{j+\frac{1}{2},-})|v'(\zeta_j)|\big|R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}}\big|, \end{aligned} \tag{2.2.27}$$

where $\tilde{\Delta}^n_{j,\pm} \coloneqq \rho^n_{j+\frac{1}{2},\pm} - \rho^n_{j-\frac{1}{2},\pm}$. Rearranging the terms of the expression (2.2.17) obtained through summation by parts, we can write

$$\begin{aligned} &R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}} \\ &\quad = (R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) - (R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}) \\ &\quad = \frac{\Delta x}{2}\Big(w^{N-1}_\eta(\rho^n_{j+N+\frac{1}{2},+} - \rho^n_{j+N-\frac{1}{2},+}) - w^0_\eta(\rho^n_{j+\frac{1}{2},+} - \rho^n_{j-\frac{1}{2},+}) \\ &\qquad + \sum_{k=1}^{N-1}(\rho^n_{j+k+\frac{1}{2},+} - \rho^n_{j+k-\frac{1}{2},+})(w^{k-1}_\eta - w^k_\eta)\Big) \\ &\qquad + \frac{\Delta x}{2}\Big(w^N_\eta(\rho^n_{j+N+\frac{3}{2},-} - \rho^n_{j+N+\frac{1}{2},-}) - w^1_\eta(\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-}) \\ &\qquad + \sum_{k=1}^{N-1}(\rho^n_{j+k+\frac{3}{2},-} - \rho^n_{j+k+\frac{1}{2},-})(w^k_\eta - w^{k+1}_\eta)\Big). \end{aligned} \tag{2.2.28}$$

Since $\zeta_j \in \mathcal{I}(R^n_{j-\frac{1}{2}}, R^n_{j+\frac{1}{2}})$, for some $\alpha, \beta \in (0,1)$ we can write

$$\begin{aligned} \zeta_{j+1} - \zeta_j &= \alpha R^n_{j+\frac{3}{2}} + (1-\alpha)R^n_{j+\frac{1}{2}} - \beta R^n_{j+\frac{1}{2}} - (1-\beta)R^n_{j-\frac{1}{2}} \\ &= \alpha\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k+\frac{3}{2},+}w^k_\eta + \rho^n_{j+k+\frac{5}{2},-}w^{k+1}_\eta\big) \\ &\qquad + (1-\alpha)\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k+\frac{1}{2},+}w^k_\eta + \rho^n_{j+k+\frac{3}{2},-}w^{k+1}_\eta\big) \\ &\qquad - \beta\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k+\frac{1}{2},+}w^k_\eta + \rho^n_{j+k+\frac{3}{2},-}w^{k+1}_\eta\big) \\ &\qquad - (1-\beta)\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k-\frac{1}{2},+}w^k_\eta + \rho^n_{j+k+\frac{1}{2},-}w^{k+1}_\eta\big) \\ &= \alpha\frac{\Delta x}{2}\sum_{k=1}^{N}\big(\rho^n_{j+k+\frac{1}{2},+}w^{k-1}_\eta + \rho^n_{j+k+\frac{3}{2},-}w^k_\eta\big) \\ &\qquad + (1-\alpha)\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k+\frac{1}{2},+}w^k_\eta + \rho^n_{j+k+\frac{3}{2},-}w^{k+1}_\eta\big) \\ &\qquad - \beta\frac{\Delta x}{2}\sum_{k=0}^{N-1}\big(\rho^n_{j+k+\frac{1}{2},+}w^k_\eta + \rho^n_{j+k+\frac{3}{2},-}w^{k+1}_\eta\big) \end{aligned}$$

$$
\begin{aligned}
&\quad - (1-\beta)\frac{\Delta x}{2}\sum_{k=-1}^{N-2}\left(\rho^n_{j+k+\frac{1}{2},+}w_\eta^{k+1} + \rho^n_{j+k+\frac{3}{2},-}w_\eta^{k+2}\right)\\
&= \frac{\Delta x}{2}(\mathcal{Q}_1 + \mathcal{Q}_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Q}_1 &:= \sum_{k=1}^{N-2}\rho^n_{j+k+\frac{1}{2},+}\Big(\alpha w_\eta^{k-1} + (1-\alpha)w_\eta^k - \beta w_\eta^k - (1-\beta)w_\eta^{k+1}\Big)\\
&\quad + \sum_{k=1}^{N-2}\rho^n_{j+k+\frac{3}{2},-}\Big(\alpha w_\eta^{k} + (1-\alpha)w_\eta^{k+1} - \beta w_\eta^{k+1} - (1-\beta)w_\eta^{k+2}\Big),\\
\mathcal{Q}_2 &:= \alpha\left(\rho^n_{j+N-\frac{1}{2},+}w_\eta^{N-2} + \rho^n_{j+N+\frac{1}{2},-}w_\eta^{N-1} + \rho^n_{j+N+\frac{1}{2},+}w_\eta^{N-1} + \rho^n_{j+N+\frac{3}{2},-}w_\eta^{N}\right)\\
&\quad + (1-\alpha)\left(\rho^n_{j+\frac{1}{2},+}w_\eta^{0} + \rho^n_{j+\frac{3}{2},-}w_\eta^{1} + \rho^n_{j+N-\frac{1}{2},+}w_\eta^{N-1} + \rho^n_{j+N+\frac{1}{2},-}w_\eta^{N}\right)\\
&\quad - \beta\left(\rho^n_{j+\frac{1}{2},+}w_\eta^{0} + \rho^n_{j+\frac{3}{2},-}w_\eta^{1} + \rho^n_{j+N-\frac{1}{2},+}w_\eta^{N-1} + \rho^n_{j+N+\frac{1}{2},-}w_\eta^{N}\right)\\
&\quad - (1-\beta)\left(\rho^n_{j-\frac{1}{2},+}w_\eta^{0} + \rho^n_{j+\frac{1}{2},-}w_\eta^{1} + \rho^n_{j+\frac{1}{2},+}w_\eta^{1} + \rho^n_{j+\frac{3}{2},-}w_\eta^{2}\right).
\end{aligned}
$$

As the function w_η is non-increasing, $w_\eta^k \geq w_\eta^{k+1}$ for each $k = 0, ..., N-1$ and it follows that

$$
\alpha w_\eta^{k-1} + (1-\alpha)w_\eta^k - \beta w_\eta^k - (1-\beta)w_\eta^{k+1} \geq 0. \tag{2.2.29}
$$

Now using (2.2.29), property (2.2.1) and property (2.2.10) of the linear reconstruction, we have the following bound

$$
\begin{aligned}
|\mathcal{Q}_1| &\leq \rho_{\max}\Bigg[\sum_{k=1}^{N-2}\Big(\alpha w_\eta^{k-1} + (1-\alpha)w_\eta^k - \beta w_\eta^k - (1-\beta)w_\eta^{k+1}\Big)\\
&\qquad + \sum_{k=1}^{N-2}\Big(\alpha w_\eta^{k} + (1-\alpha)w_\eta^{k+1} - \beta w_\eta^{k+1} - (1-\beta)w_\eta^{k+2}\Big)\Bigg]\\
&\leq \rho_{\max}\Big[\Big(\alpha w_\eta^0 + (1-\beta)w_\eta^1 - \alpha w_\eta^{N-2} - (1-\beta)w_\eta^{N-1}\Big)\\
&\qquad + \Big(\alpha w_\eta^1 + (1-\beta)w_\eta^2 - \alpha w_\eta^{N-1} - (1-\beta)w_\eta^{N}\Big)\Big]\\
&\leq 4w_\eta^0\rho_{\max}.
\end{aligned}
$$

In a similar way, using property (2.2.1) and since $0 < \alpha, \beta < 1$, we obtain the following bound for Q_2

$$
|\mathcal{Q}_2| \leq 16w_\eta^0\rho_{\max}.
$$

Thus, we have

$$
|\zeta_{j+1} - \zeta_j| \leq \frac{\Delta x}{2}\big(|\mathcal{Q}_1| + |\mathcal{Q}_2|\big) \leq 10\Delta x w_\eta^0\rho_{\max}. \tag{2.2.30}
$$

Now, from (2.2.27) we write

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\rho^{n+1}_{j+1}-\rho^{n+1}_{j}| \le{}& \frac{1}{2}\sum_{j\in\mathbb{Z}}|\tilde{\Delta}^n_{j,+}| + \sum_{j\in\mathbb{Z}}|\tilde{\Delta}^n_{j+1,-}|\left(\frac{1}{2}-\lambda V^n_{j+\frac{3}{2}}g'(\zeta_{j+1,-})\right)\\ &+\sum_{j\in\mathbb{Z}}\lambda V^n_{j-\frac{1}{2}}g'(\zeta_{j,-})|\tilde{\Delta}^n_{j,-}|\\ &+\sum_{j\in\mathbb{Z}}\lambda g(\rho^n_{j+\frac{1}{2},-})|v''(\zeta_{j+\frac{1}{2}})||\zeta_{j+1}-\zeta_j||R^n_{j+\frac{3}{2}}-R^n_{j+\frac{1}{2}}|\\ &+\sum_{j\in\mathbb{Z}}\lambda g(\rho^n_{j+\frac{1}{2},-})|v'(\zeta_j)||R^n_{j+\frac{3}{2}}-2R^n_{j+\frac{1}{2}}+R^n_{j-\frac{1}{2}}|\\ \le{}& \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4,\end{aligned}$$

where

$$\mathcal{A}_1 := \frac{1}{2}\sum_{j\in\mathbb{Z}}|\tilde{\Delta}^n_{j,+}|,\quad \mathcal{A}_2 := \sum_{j\in\mathbb{Z}}|\tilde{\Delta}^n_{j+1,-}|\left(\frac{1}{2}+\lambda(V^n_{j+\frac{1}{2}}-V^n_{j+\frac{3}{2}})g'(\zeta_{j+1,-})\right),$$

$$\mathcal{A}_3 := \sum_{j\in\mathbb{Z}}\lambda g(\rho^n_{j+\frac{1}{2},-})|v''(\zeta_{j+\frac{1}{2}})||\zeta_{j+1}-\zeta_j|\left(\sum_{k=0}^{N-1}\frac{\Delta x}{2}w^k_\eta|\tilde{\Delta}^n_{j+k+1,+}|+\sum_{k=0}^{N-1}\frac{\Delta x}{2}w^{k+1}_\eta|\tilde{\Delta}^n_{j+k+2,-}|\right),$$

$$\begin{aligned}\mathcal{A}_4 := \sum_{j\in\mathbb{Z}}\lambda g(\rho^n_{j+\frac{1}{2},-})|v'(\zeta_j)|&\left[\frac{\Delta x}{2}\left(w^{N-1}_\eta|\tilde{\Delta}_{j+N,+}|+w^0_\eta|\tilde{\Delta}_{j,+}|+\sum_{k=1}^{N-1}|\tilde{\Delta}^n_{j+k,+}|(w^{k-1}_\eta-w^k_\eta)\right)\right.\\ &\left.+\frac{\Delta x}{2}\left(w^N_\eta|\tilde{\Delta}^n_{j+N+1,-}|+w^1_\eta|\tilde{\Delta}^n_{j+1,-}|+\sum_{k=1}^{N-1}|\tilde{\Delta}^n_{j+k+1,-}|(w^k_\eta-w^{k+1}_\eta)\right)\right].\end{aligned}$$

Note that $\mathcal{A}_2$ is obtained by shifting the index and grouping the second and third terms in (2.2.27), $\mathcal{A}_3$ is obtained using (2.2.16) and $\mathcal{A}_4$ using (2.2.28). The property (2.2.11) of the reconstruction reads as $\sum_{j\in\mathbb{Z}}|\tilde{\Delta}^n_{j,\pm}| \le \mathrm{TV}(\rho^n_{\Delta x})$. Consequently, using the estimate (2.2.20) the following bound holds

$$|\mathcal{A}_2| \le \left(\frac{1}{2}+\Delta t\|v'\|w^0_\eta\|g'\|\rho_{\max}\right)\mathrm{TV}(\rho^n_{\Delta x}).$$

Further, using the estimate (2.2.30), property (2.2.11) and using the fact that $0 \le \sum_{k=0}^{N-1}\Delta x w^k_\eta \le w^0_\eta N\Delta x = w^0_\eta\eta$, we obtain

$$|\mathcal{A}_3| \le 10\rho_{\max}\Delta t\|g\|\|v''\|(w^0_\eta)^2\eta\mathrm{TV}(\rho^n_{\Delta x}).$$

By using property (2.2.11) and observing that $0 \le \sum_{k=1}^{N-1}(w^{k-1}_\eta - w^k_\eta) = w^0_\eta - w^{N-1}_\eta \le w^0_\eta$ and $0 \le \sum_{k=1}^{N-1}(w^k_\eta - w^{k+1}_\eta) = w^1_\eta - w^N_\eta \le w^1_\eta \le w^0_\eta$, we have the bound $|\mathcal{A}_4| \le 3\|g\|\|v'\|w^0_\eta\mathrm{TV}(\rho^n_{\Delta x})$. Finally, we can write

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}|\rho^{n+1}_{j+1}-\rho^{n+1}_j|\\ &\le \mathrm{TV}(\rho^n_{\Delta x})\left(1+\Delta t\Big(w^0_\eta\rho_{\max}\|v'\|\|g'\|+10\rho_{\max}(w^0_\eta)^2\|v''\|\|g\|\eta+3w^0_\eta\|g\|\|v'\|\Big)\right).\end{aligned}$$

Hence we obtain the desired bound on the total variation as

$$\mathrm{TV}(\rho_{\Delta x}^{n+1}) \leq (1 + C\Delta t)\mathrm{TV}(\rho_{\Delta x}^{n}),$$

where $C = w_\eta^0 \rho_{\max} \|v'\| \|g'\| + 10\rho_{\max}(w_\eta^0)^2 \|v''\| \|g\| \eta + 3w_\eta^0 \|g\| \|v'\|$. $\square$

Theorem 2.2.7. *(BV estimate in space) Let the initial data $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$ and the CFL condition (2.2.13) holds. Then for every $T > 0$ the approximate solution $\rho_{\Delta x}$ obtained using the second-order scheme (2.2.8) satisfies the space total variation estimate*

$$\mathrm{TV}\big(\rho_{\Delta x}(T, \cdot)\big) \leq \exp(2TC)\mathrm{TV}(\rho_0), \tag{2.2.31}$$

where $C := w_\eta^0 \rho_{\max} \|v'\| \|g'\| + 10\rho_{\max}(w_\eta^0)^2 \|v''\| \|g\| \eta + 3w_\eta^0 \|g\| \|v'\|$.

Proof. Let $\{\rho_j^{n+1}\}_{j\in\mathbb{Z}}$ be calculated using the second-order scheme (2.2.8). We can write

$$\mathrm{TV}(\rho_{\Delta x}^{n+1}) = \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n+1} - \rho_j^{n+1}| \leq \frac{1}{2}\Big(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n} - \rho_j^{n}| + \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{(2)} - \rho_j^{(2)}|\Big).$$

Applying Lemma 2.2.6 on the two Euler forward steps in (2.2.8) we get the following bound,

$$\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{(2)} - \rho_j^{(2)}| \leq (1 + C\Delta t) \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{(1)} - \rho_j^{(1)}| \leq (1 + C\Delta t)^2 \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n} - \rho_j^{n}|.$$

Therefore,

$$\begin{aligned}
\mathrm{TV}(\rho_{\Delta x}^{n+1}) &\leq \frac{1}{2}\Big(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n} - \rho_j^{n}| + (1 + C\Delta t)^2 \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n} - \rho_j^{n}|\Big) \\
&\leq (1 + C\Delta t)^2 \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n} - \rho_j^{n}| \\
&\leq (1 + C\Delta t)^{2(n+1)} \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{0} - \rho_j^{0}| \\
&\leq \exp\big(2\Delta t(n+1)C\big)\mathrm{TV}(\rho_0) \leq \exp(2TC)\mathrm{TV}(\rho_0),
\end{aligned}$$

whenever $(n+1)\Delta t \leq T$. Thus we have $\mathrm{TV}\big(\rho_{\Delta x}(T, \cdot)\big) \leq \exp(2TC)\mathrm{TV}(\rho_0)$. $\square$

2.2.3 L^1- Lipschitz continuity in time

Lemma 2.2.8. *Let $\rho_{\Delta x}^n \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$ be the piecewise constant function given by $\rho_{\Delta x}^n(x) = \rho_j^n$ for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. If $\rho_{\Delta x}^{n+1}$ is computed using the Euler forward step (2.2.14) with the CFL condition (2.2.13), then the following estimate holds*

$$\|\rho_{\Delta x}^{n+1} - \rho_{\Delta x}^{n}\|_{\mathrm{L}^1(\mathbb{R})} \leq \Delta t\Big(\|g\| \|v'\| w_\eta^0 \eta + \|g'\| \|v\|\Big)\mathrm{TV}(\rho_{\Delta x}^{n}). \tag{2.2.32}$$

Proof. From (2.2.14) we write

$$\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} = \Delta x \sum_{j\in\mathbb{Z}} |\rho^{n+1}_j - \rho^n_j| = \Delta t \sum_{j\in\mathbb{Z}} |\bar{F}^n_{j-\frac{1}{2}} - \bar{F}^n_{j+\frac{1}{2}}|,$$

where $\bar{F}_{j+\frac{1}{2}} \coloneqq g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}}$. Subtracting and adding $g(\rho^n_{j-\frac{1}{2},-})V^n_{j+\frac{1}{2}}$ to the term $\bar{F}^n_{j-\frac{1}{2}} - \bar{F}^n_{j+\frac{1}{2}}$ and using the mean value theorem, we deduce that

$$\begin{aligned}
\left|\bar{F}^n_{j-\frac{1}{2}} - \bar{F}^n_{j+\frac{1}{2}}\right| &= \left|g(\rho^n_{j-\frac{1}{2},-})(V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}}) + \big(g(\rho^n_{j-\frac{1}{2},-}) - g(\rho^n_{j+\frac{1}{2},-})\big)V^n_{j+\frac{1}{2}}\right| \\
&= \left|-v'(\zeta_j)g(\rho^n_{j+\frac{1}{2},-})(R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}) + g'(\theta_{j,-})\big(\rho^n_{j-\frac{1}{2},-} - \rho^n_{j+\frac{1}{2},-}\big)V^n_{j+\frac{1}{2}}\right| \\
&\leq \|g\|\|v'\|\frac{\Delta x}{2}\sum_{k=0}^{N-1}\Big(w^k_\eta|\rho^n_{j+k+\frac{1}{2},+} - \rho^n_{j+k-\frac{1}{2},+}| + w^{k+1}_\eta|\rho^n_{j+k+\frac{3}{2},-} - \rho^n_{j+k+\frac{1}{2},-}|\Big) \\
&\quad + \|g'\|\|v\||\rho^n_{j-\frac{1}{2},-} - \rho^n_{j+\frac{1}{2},-}|,
\end{aligned}$$

for $\zeta_j \in \mathcal{I}(R^n_{j-\frac{1}{2}}, R^n_{j+\frac{1}{2}})$ and $\theta_{j,-} \in \mathcal{I}(\rho^n_{j-\frac{1}{2},-}, \rho^n_{j+\frac{1}{2},-})$. Further, invoking the properties (2.2.11) and (2.2.2) yields

$$\begin{aligned}
&\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} \\
&= \Delta t \sum_{j\in\mathbb{Z}}|\bar{F}^n_{j-\frac{1}{2}} - \bar{F}^n_{j+\frac{1}{2}}| \\
&\leq \Delta t \Bigg[\|g\|\|v'\|\frac{\Delta x}{2}\sum_{k=0}^{N-1}\Bigg(w^k_\eta\sum_{j\in\mathbb{Z}}|\rho^n_{j+k+\frac{1}{2},+} - \rho^n_{j+k-\frac{1}{2},+}| + w^{k+1}_\eta\sum_{j\in\mathbb{Z}}|\rho^n_{j+k+\frac{3}{2},-} - \rho^n_{j+k+\frac{1}{2},-}|\Bigg) \\
&\qquad + \|g'\|\|v\|\sum_{j\in\mathbb{Z}}|\rho^n_{j-\frac{1}{2},-} - \rho^n_{j+\frac{1}{2},-}|\Bigg] \\
&\leq \Delta t\left(\|g\|\|v'\|\mathrm{TV}(\rho^n_{\Delta x})w^0_\eta N\Delta x + \|g'\|\|v\|\mathrm{TV}(\rho^n_{\Delta x})\right) \\
&= \Delta t\left(\|g\|\|v'\|w^0_\eta\eta + \|g'\|\|v\|\right)\mathrm{TV}(\rho^n_{\Delta x}).
\end{aligned}$$

□

Theorem 2.2.9. *(*L^1*- Lipschitz continuity in time) Let* $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ *and the CFL condition* (2.2.13) *holds. Then the approximate solution constructed using the second-order scheme* (2.2.8) *is an* L^1*- Lipschitz continuous function of time, i.e, for any* $T > 0$*, there exists a constant* κ_T *such that*

$$\|\rho_{\Delta x}(t,\cdot) - \rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq \kappa_T(|t-s| + \Delta t) \quad \text{for } t,s \in [0,T]. \tag{2.2.33}$$

Proof. For the second-order scheme (2.2.8), we see that

$$\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} = \Delta x\sum_{j\in\mathbb{Z}}\left|\frac{\rho^n_j + \rho^{(2)}_j}{2} - \rho^n_j\right| = \frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}|\rho^{(2)}_j - \rho^n_j|. \tag{2.2.34}$$

Upon subtracting and adding $\rho_j^{(1)}$ to the term $\rho_j^{(2)} - \rho_j^n$, employing Lemma 2.2.8 together with Lemma 2.2.6 and subsequently using Theorem 2.2.7, it follows from (2.2.34) that

$$
\begin{aligned}
&\|\rho_{\Delta x}^{n+1} - \rho_{\Delta x}^n\|_{\mathrm{L}^1(\mathbb{R})} \\
&\leq \frac{\Delta x}{2}\left(\sum_{j\in\mathbb{Z}} |\rho_j^{(2)} - \rho_j^{(1)}| + \sum_{j\in\mathbb{Z}} |\rho_j^{(1)} - \rho_j^n|\right) \\
&\leq \frac{\Delta t}{2}\Big(\mathrm{TV}\big(\rho_{\Delta x}^{(1)}(t^n,\cdot)\big)(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|) + \mathrm{TV}\big(\rho_{\Delta x}^n\big)(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|)\Big) \\
&\leq \frac{\Delta t}{2}(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|)\left((1+C\Delta t)\mathrm{TV}\big(\rho_{\Delta x}^n\big) + \mathrm{TV}\big(\rho_{\Delta x}^n\big)\right) \\
&\leq \frac{\Delta t}{2}(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|)\mathrm{TV}\big(\rho_{\Delta x}^n\big)\Big(2 + C\Delta t\Big) \\
&\leq \Delta t(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|)\exp(2TC)\mathrm{TV}(\rho_0)\left(1 + C\frac{\Delta t}{2}\right),
\end{aligned}
$$

provided that the CFL condition (2.2.13) holds. Furthermore, we can write

$$
\begin{aligned}
\|\rho_{\Delta x}^m - \rho_{\Delta x}^n\|_{\mathrm{L}^1(\mathbb{R})} &\leq \|\rho_{\Delta x}^m - \rho_{\Delta x}^{m-1}\|_{\mathrm{L}^1(\mathbb{R})} + \cdots + \|\rho_{\Delta x}^{n+1} - \rho_{\Delta x}^n\|_{\mathrm{L}^1(\mathbb{R})} \\
&\leq \Delta t\tilde{\kappa}_T + \cdots + \Delta t\tilde{\kappa}_T = \tilde{\kappa}_T|n-m|\Delta t,
\end{aligned}
$$

with $\tilde{\kappa}_T = \big(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|\big)\exp(2TC)\mathrm{TV}(\rho_0)\Big(1 + \frac{\Delta t}{2}C\Big)$ and for $m, n \in \mathbb{N}, m > n$ with $m\Delta t < T$ and $n\Delta t < T$. Thus we can conclude that for sufficiently small Δt,

$$
\|\rho_{\Delta x}^n - \rho_{\Delta x}^m\|_{\mathrm{L}^1(\mathbb{R})} \leq \kappa_T|n-m|\Delta t \tag{2.2.35}
$$

for $n, m \in \mathbb{N}$ with $n\Delta t < T$ and $m\Delta t < T$, where

$$
\kappa_T = \big(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|\big)\exp(2TC)\mathrm{TV}(\rho_0)(1 + C).
$$

Now, for $t, s \in [0,T]$, let n, m be such that $t \in [t^n, t^{n+1})$ and $s \in [t^m, t^{m+1})$. Since $|n-m|\Delta t \leq |t-s| + \Delta t$, from (2.2.35) it follows that

$$
\|\rho_{\Delta x}(t,\cdot) - \rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq \kappa_T(|t-s| + \Delta t) \quad \text{for } t, s \in [0,T]. \tag{2.2.36}
$$

□

2.3 Convergence to a weak solution

The results in Theorems 2.2.4, 2.2.7 and 2.2.9 allow us to use the Kolmogorov's theorem (as described in Theorem A.8 of [108]) to extract a convergence subsequence of approximate solutions obtained using the second-order scheme (2.2.8). We have adapted the Kolmogorov's theorem to fit our specific context. For the sake of completeness, we state the modified theorem here and the proof is given in the Appendix A.1. Further, we use a Lax-Wendroff type argument to show that the limit is a weak solution.

Theorem 2.3.1. *Let $u_\xi : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ be a family of functions such that for each positive T,*

$$|u_\xi(t,x)| \leq \mu_T \tag{2.3.1}$$

for $(t,x) \in [0,T) \times \mathbb{R}$ and a constant μ_T independent of ξ. Assume in addition for all compact set $B \subset \mathbb{R}$ and for $t \in [0,T]$ that

$$\sup_{|\zeta|\leq|\tau|} \int_B |u_\xi(t,x+\zeta) - u_\xi(t,x)| \,\mathrm{d}x \leq \nu_T^B |\tau|, \tag{2.3.2}$$

for a modulus of continuity ν_T^B . Furthermore, assume for s and t in $[0,T]$ that

$$\int_B |u_\xi(t,x) - u_\xi(s,x)| \,\mathrm{d}x \leq \omega_T^B(|t-s|) + \mathcal{O}(\xi), \tag{2.3.3}$$

as $\xi \to 0$ for some modulus of continuity ω_T^B. Then there exists a sequence $\xi_j \to 0$ such that for each $t \in [0,T]$ the sequence $\{u_{\xi_j}(t,\cdot)\}$ converges to a function $u(t,\cdot)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, the convergence is in $\mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$.

Now, in the following theorem we establish the convergence of a subsequence of the approximate solutions to a weak solution of the problem (2.1.1).

Theorem 2.3.2. *(Convergence to a weak solution) Let $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ and let $\rho_{\Delta x}$ be the approximate solution obtained using the second-order scheme* (2.2.8) *under the CFL condition* (2.2.13)*. Then corresponding to any sequence $\Delta x_k \to 0$, there exists a subsequence, still denoted by Δx_k, such that $\rho_{\Delta x_k}$ converges in $\mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ to a weak solution of (2.1.1).*

Proof. Firstly, the existence of a convergent subsequence is proven by using the Kolmogorov's theorem 2.3.1 invoking the estimates derived in Theorems 2.2.4, 2.2.7 and 2.2.9 given by

$$\|\rho_{\Delta x}\| \leq \|\rho_0\|, \tag{2.3.4}$$

$$\mathrm{TV}(\rho_{\Delta x}(t,\cdot)) \leq \exp(2TC)\mathrm{TV}(\rho_0) \quad \text{for } t \in [0,T] \tag{2.3.5}$$

and

$$\|\rho_{\Delta x}(t,\cdot) - \rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq \kappa_T(|t-s| + \Delta t) \quad \text{for } t,s \in [0,T], \tag{2.3.6}$$

respecitvely. Under the CFL conditon (2.2.13), the family $\{\rho_{\Delta x}\}$ obtained from the second-order scheme (2.2.8) satisfies (2.3.1) with $\mu_T = \|\rho_0\|$. By Lemma A.1 of [108], the total variation bound (2.3.5) ensures that the family satisfies (2.3.2) with $\nu_T^B = \exp(2TC)\mathrm{TV}(\rho_0)$.

Additionally, using (2.3.6), we observe that the family $\{\rho_{\Delta x}\}$ satisfies (2.3.3) with $\omega_T^B = \kappa_T$. Now by Theorem 2.3.1, corresponding to any sequence $\Delta x_k \to 0$, there exists a subsequence, still denoted by Δx_k, such that $\rho_{\Delta x_k}$ converges to a function ρ in $\mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and consequently in $\mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$.

Our next step is to show that the limit ρ is a weak solution of (2.1.1). Typically, we will use a Lax-Wendroff type argument [128], with certain modifications to deal with the numerical flux which also depends on Δx. Denote the convergent subsequence obtained above by $\rho_{\Delta x}$. Let $\varphi \in \mathrm{C}^1_c([0,T) \times \mathbb{R})$. Let T_φ be such that $0 \le T_\varphi < T$ and $\varphi(t,x) = 0$ for $t \ge T_\varphi$ and let n_T be such that $T_\varphi \in (n_T\Delta t, (n_T+1)\Delta t]$. Multiplying the conservative form (2.2.9) by $\varphi(t^n, x_j)$ and summing over n and j yields

$$\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \varphi(t^n,x_j)(\rho_j^{n+1} - \rho_j^n) = -\lambda \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \varphi(t^n,x_j)(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}).$$

Further, summing by parts we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \varphi(0,x_j)\rho_j^0 + \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \Big(\varphi(t^{n+1},x_j) - \varphi(t^n,x_j)\Big)\rho_j^{n+1}& \\ + \lambda \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} F^n_{j+\frac{1}{2}}\Big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\Big) &= 0.\end{aligned}$$

Now, multiplying the above expression by Δx we see that

$$\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = 0, \tag{2.3.7}$$

where we define the terms

$$\begin{aligned}\mathcal{P}_1 &\coloneqq \Delta x \sum_{j\in\mathbb{Z}} \varphi(0,x_j)\rho_j^0, \quad \mathcal{P}_2 \coloneqq \Delta t \Delta x \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \frac{(\varphi(t^{n+1},x_j) - \varphi(t^n,x_j))}{\Delta t}\rho_j^{n+1}, \\ \mathcal{P}_3 &\coloneqq \Delta t \Delta x \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} F^n_{j+\frac{1}{2}} \frac{(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j))}{\Delta x}.\end{aligned}$$

We can also write

$$\begin{aligned}\mathcal{P}_1 + \mathcal{P}_2 = &\int_{-\infty}^{+\infty} \rho_{\Delta x}(0,x)\varphi_{\Delta x}(0,x)\,\mathrm{d}x \\ &+ \int_{t=0}^{T}\int_{-\infty}^{+\infty} \rho_{\Delta x}(t+\Delta t,x)\partial_t\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.3.8}$$

where

$$\begin{aligned}\varphi_{\Delta x}(0,x) &\coloneqq \varphi(0,x_j) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \\ \partial_t\varphi_{\Delta x}(t,x) &\coloneqq \varphi_t(\bar{t},x_j) \quad \text{for } t \in [t^n,t^{n+1}), x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \text{ for some } \bar{t} \in (t^n,t^{n+1}).\end{aligned}$$

By the dominated convergence theorem it follows that

$$\lim_{\Delta x\to 0}(\mathcal{P}_1+\mathcal{P}_2)=\int_{-\infty}^{+\infty}\rho(0,x)\varphi(0,x)\,\mathrm{d}x+\int_{t=0}^{T}\int_{-\infty}^{+\infty}\rho(t,x)\varphi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

We define $R_j^n:=\Delta x\sum_{k=0}^{N-1}\hat{w}_\eta^k\rho_{j+k}^n$, $R_j^{(1)}:=\Delta x\sum_{k=0}^{N-1}\hat{w}_\eta^k\rho_{j+k}^{(1)}$, $V_j^n:=v(R_j^n)$ and $V_j^{(1)}:=v(R_j^{(1)})$, where $\hat{w}_\eta^k:=\dfrac{w_\eta^k}{\hat{Q}_{\Delta x}}$, and $\hat{Q}_{\Delta x}:=\Delta x\displaystyle\sum_{k=0}^{N-1}w_\eta^k$. Note that $\hat{Q}_{\Delta x}\approx\int_0^\eta w_\eta(y)\,\mathrm{d}y=1$. Further, there exists a constant $L>0$ such that $\left|\dfrac{\hat{Q}_{\Delta x}-1}{\hat{Q}_{\Delta x}}\right|\le L\Delta x$ for sufficiently small Δx. Here we use the modified weights $\hat{w}_\eta^k$ to ensure that R_j^n and $R_j^{(1)}$ fall in the range $[0,\rho_{\max}]$. By adding and subtracting $\frac{1}{2}\Delta t\big(g(\rho_j^n)V_j^n+g(\rho_j^{(1)})V_j^{(1)}\big)\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)$ into the term $\mathcal{P}_3$ in (2.3.7), it reads as

$$\mathcal{P}_3=\mathcal{S}_1+\mathcal{S}_2, \tag{2.3.9}$$

where we define

$$\begin{aligned}\mathcal{S}_1&:=\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\left(F_{j+\frac{1}{2}}^n-\frac{g(\rho_j^n)V_j^n+g(\rho_j^{(1)})V_j^{(1)}}{2}\right)\frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x},\\ \mathcal{S}_2&:=\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\left(\frac{g(\rho_j^n)V_j^n+g(\rho_j^{(1)})V_j^{(1)}}{2}\right)\frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x}.\end{aligned}$$

Further, by the mean value theorem we observe that

$$\begin{aligned}g(\rho_j^{(1)})&=g\left(\rho_j^n-\lambda\big(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n-g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\big)\right)\\&=g(\rho_j^n)-\lambda g'(\zeta_j)\left(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n-g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\right)\end{aligned} \tag{2.3.10}$$

for some $\zeta_j\in\mathcal{I}(\rho_j^n,\rho_j^{(1)})$. Similarly,

$$\begin{aligned}V_j^{(1)}&=v\left(\Delta x\sum_{k=0}^{N-1}\hat{w}_\eta^k\rho_{j+k}^{(1)}\right)\\&=v\left(\Delta x\sum_{k=0}^{N-1}\hat{w}_\eta^k\rho_{j+k}^n-\Delta t\sum_{k=0}^{N-1}\hat{w}_\eta^k\Big(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n-g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\Big)\right)\\&=v\left(\Delta x\sum_{k=0}^{N-1}\hat{w}_\eta^k\rho_{j+k}^n\right)-\Delta t v'(\theta_j)\sum_{k=0}^{N-1}\hat{w}_\eta^k\left(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n-g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\right)\\&=V_j^n-\Delta t v'(\theta_j)\sum_{k=0}^{N-1}\hat{w}_\eta^k\left(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n-g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\right),\end{aligned}$$

for some $\theta_j\in\mathcal{I}(R_j^n,R_j^{(1)})$. As a result, the term $\mathcal{S}_2$ in (2.3.9) can be written as

$$\mathcal{S}_2=\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4, \tag{2.3.11}$$

where

$$\begin{aligned}
\mathcal{T}_1 &\coloneqq \Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} g(\rho_j^n) V_j^n \frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x},\\
\mathcal{T}_2 &\coloneqq -\frac{1}{2}\Delta t^2 \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} g(\rho_j^n) v'(\theta_j) \ell(n,j,k) \frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x},\\
\mathcal{T}_3 &\coloneqq -\frac{1}{2}\Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \lambda V_j^n g'(\zeta_j) q(n,j) \frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x},\\
\mathcal{T}_4 &\coloneqq \frac{1}{2}\Delta t^2 \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} \lambda g'(\zeta_j) q(n,j) v'(\theta_j) \ell(n,j,k) \frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x},
\end{aligned}$$

with the definition $\ell(n,j,k) \coloneqq \hat{w}_\eta^k \left(g(\rho^n_{j+k+\frac{1}{2},-}) V^n_{j+k+\frac{1}{2}} - g(\rho^n_{j+k-\frac{1}{2},-}) V^n_{j+k-\frac{1}{2}} \right)$ and $q(n,j) \coloneqq \left(g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right)$. Note that, the term $\mathcal{T}_1$ can also be written as

$$\mathcal{T}_1 = \int_0^T \int_{-\infty}^{+\infty} g\left(\rho_{\Delta x}(t,x)\right) v\left(R_{\Delta x}(t,x)\right) \partial_x \varphi_{\Delta x}(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$

where

$$\begin{aligned}
R_{\Delta x}(t,x) &\coloneqq R_j^n \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}),\\
\partial_x \varphi_{\Delta x}(t,x) &\coloneqq \varphi_x(t^n, \bar{x}) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}],\ t \in [t^n, t^{n+1}) \text{ for some } \bar{x} \in (x_j, x_{j+1}).
\end{aligned}$$

Note that

$$R_{\Delta x}(t,x) = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta} \rho_{\Delta x}(t,y) w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}}) \,\mathrm{d}y \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}),$$

where $w_{\eta,\Delta x}(x) \coloneqq \hat{w}_\eta^k$ for $x \in (k\Delta x, (k+1)\Delta x]$. By the dominated convergence theorem, it is clear that $R_{\Delta x}(t,x)$ converges to $\int_x^{x+\eta} \rho(t,y) w_\eta(y-x) \,\mathrm{d}y$ as $\Delta x \to 0$. Now, applying the dominated convergence theorem again, we have

$$\lim_{\Delta x \to 0} \mathcal{T}_1 = \int_0^T \int_{-\infty}^{+\infty} g\big(\rho(t,x)\big) v\big(\rho * w_\eta(t,x)\big) \varphi_x(t,x) \,\mathrm{d}x \,\mathrm{d}t.$$

We will now show that the terms $\mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$ in (2.3.11) tend to 0 as $\Delta x \to 0$. To proceed further, we consider the following fact

$$\begin{aligned}
&\big| g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \big| \\
&\quad \le \|g\| \|v'\| \Delta x w_\eta^0 \rho_{\max} + \|v\| \|g'\| |\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}| \\
&\quad \le \|g\| \|v'\| \Delta x w_\eta^0 \rho_{\max} + 2\|v\| \|g'\| |\rho^n_j - \rho^n_{j-1}|,
\end{aligned} \tag{2.3.12}$$

which is obtained by writing

$$g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}}$$

$$
\begin{aligned}
&= g(\rho^n_{j+\frac{1}{2},-})(V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) + V^n_{j-\frac{1}{2}}\big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_{j-\frac{1}{2},-})\big) \\
&= g(\rho^n_{j+\frac{1}{2},-})(V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) + V^n_{j-\frac{1}{2}} g'(\zeta_{j,-})(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}),
\end{aligned}
$$

for some $\zeta_{j,-} \in \mathcal{I}(\rho^n_{j-\frac{1}{2},-}, \rho^n_{j+\frac{1}{2},-})$ and using (2.2.20) and property (2.2.12). Now, using summation by parts, the term in $\mathcal{T}_2$ can be reformulated as

$$
\begin{aligned}
\sum_{k=0}^{N-1} \ell(n,j,k) = \hat{w}_\eta^{N-1} g(\rho^n_{j+N-\frac{1}{2},-}) V^n_{j+N-\frac{1}{2}} - \hat{w}^0_\eta g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \\
+ \sum_{k=1}^{N-1} g(\rho^n_{j+k-\frac{1}{2},-}) V^n_{j+k-\frac{1}{2}} (\hat{w}_\eta^{k-1} - \hat{w}^k_\eta).
\end{aligned}
$$

Taking absolute values and using property (2.2.1), we deduce that

$$
\begin{aligned}
\left|\sum_{k=0}^{N-1} \ell(n,j,k)\right| &\leq 2\hat{w}^0_\eta \|g\|\|v\| + \|g\|\|v\| \sum_{k=1}^{N-1} (\hat{w}_\eta^{k-1} - \hat{w}^k_\eta) \\
&\leq 2\hat{w}^0_\eta \|g\|\|v\| + \|g\|\|v\|\hat{w}^0_\eta \leq 3\hat{w}^0_\eta \|g\|\|v\|.
\end{aligned}
\tag{2.3.13}
$$

Let $R > 0$ be such that $\varphi(t,x) = 0$ for $|x| > R$. Let $j_0, j_1 \in \mathbb{Z}$ such that $-R \in (x_{j_0-\frac{1}{2}}, x_{j_0+\frac{1}{2}}]$ and $R \in (x_{j_1-\frac{1}{2}}, x_{j_1+\frac{1}{2}}]$. By using the estimate (2.3.13) and the mean value theorem, we obtain a bound on the term $\mathcal{T}_2$ as

$$
\begin{aligned}
|\mathcal{T}_2| &\leq \frac{1}{2}\Delta t^2 \Delta x \|g\|\|v'\|\|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \left|\sum_{k=0}^{N-1} \ell(n,j,k)\right| \\
&\leq \frac{3}{2}\hat{w}^0_\eta \Delta t \|g\|^2 \|v'\|\|v\|\|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \Delta t \Delta x \leq 3\hat{w}^0_\eta \Delta t \|g\|^2 \|v'\|\|v\|\|\varphi_x\| RT.
\end{aligned}
$$

Additionally, using (2.3.12) and Theorem 2.2.7, the term $\mathcal{T}_3$ in (2.3.11) can be bounded as

$$
\begin{aligned}
|\mathcal{T}_3| &\leq \frac{1}{2}\Delta t \Delta x \|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \lambda V^n_j g'(\zeta_j) \left| g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right| \\
&\leq \frac{1}{2}\Delta t \Delta x \|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \lambda V^n_j g'(\zeta_j) \left(\|g\|\|v'\|\Delta x w^0_\eta \rho_{\max} + 2\|v\|\|g'\||\rho^n_j - \rho^n_{j-1}| \right) \\
&\leq \frac{1}{2}\lambda \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\|\Delta x w^0_\eta \rho_{\max} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \Delta t \Delta x \\
&\quad + \Delta t \Delta x \lambda \|\varphi_x\|\|v\|^2 \|g'\|^2 \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} |\rho^n_j - \rho^n_{j-1}| \\
&\leq \frac{1}{2}\Delta t \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\| w^0_\eta \rho_{\max} 2RT \\
&\quad + \lambda \|\varphi_x\|\|v\|^2 \|g'\|^2 \Delta x \int_0^T \int_{-R}^{R} \frac{|\rho_{\Delta x}(t,x) - \rho_{\Delta x}(t, x-\Delta x)|}{\Delta x} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Delta t \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\| w^0_\eta \rho_{\max} RT + \Delta t \|\varphi_x\|\|v\|^2 \|g'\|^2 \exp(2TC)\mathrm{TV}(\rho_0) T,
\end{aligned}
$$

where C is as given in Theorem 2.2.7. Furthermore, using the Theorem 2.2.7 and the estimates (2.3.12) and (2.3.13), we obtain a bound for the term $\mathcal{T}_4$ in (2.3.11) as

$$\begin{aligned}
|\mathcal{T}_4| \leq & \frac{1}{2}\Delta t^2 \Delta x \sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1} \lambda \|g'\|\|v'\| \left(\|g\|\|v'\|\Delta x w_\eta^0 \rho_{\max}\right. \\
& \left. +2\|v\|\|g'\||\rho_j^n - \rho_{j-1}^n|\right) 3\hat{w}_\eta^0 \|g\|\|v\|\|\varphi_x\| \\
\leq & \frac{3}{2}\Delta t \lambda \|g'\|\|v'\| \left(\|g\|\|v'\|\Delta x w_\eta^0 \rho_{\max}\right) \hat{w}_\eta^0 \|g\|\|v\|\|\varphi_x\| 2RT \\
& + 3\Delta t \lambda \|g'\|\|v'\|\|v\|\|g'\|\hat{w}_\eta^0\|g\|\|v\|\|\varphi_x\|\Delta x \int_{-R}^{R}\int_0^T \frac{|\rho_{\Delta x}(t,x) - \rho_{\Delta x}(t, x-\Delta x)|}{\Delta x}\,\mathrm{d}x\,\mathrm{d}t \\
\leq & \frac{3}{2}\Delta t^2 \|g'\|\|v'\|^2\|g\|^2\|v\| w_\eta^0 \hat{w}_\eta^0 \|\varphi_x\| 2RT\rho_{\max} \\
& + 3\Delta t^2 \|g'\|^2 \|v'\|\|g\|\|v\|^2 \hat{w}_\eta^0 \|\varphi_x\| \exp(2TC)\mathrm{TV}(\rho_0) T.
\end{aligned}$$

Thus, from the estimates obtained for the terms $\mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$, we can conclude that

$$\lim_{\Delta x \to 0} \mathcal{T}_2 = \lim_{\Delta x \to 0} \mathcal{T}_3 = \lim_{\Delta x \to 0} \mathcal{T}_4 = 0.$$

Therefore,

$$\begin{aligned}
\lim_{\Delta x \to 0} \mathcal{S}_2 &= \lim_{\Delta x \to 0} \Big(\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4\Big) \\
&= \int_0^T \int_{-\infty}^{+\infty} g\big(\rho(t,x)\big) v\big(\rho * w_\eta(t,x)\big) \varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.3.14}$$

Finally, we show that the term $\mathcal{S}_1$ in (2.3.9) converges to 0 as $\Delta x \to 0$. Now, using the form of $F^n_{j+\frac{1}{2}}$ in (2.2.9), the term $\mathcal{S}_1$ can be expressed as

$$\mathcal{S}_1 = \Delta t \Delta x \sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1} (\mathcal{D}_1 + \mathcal{D}_2) \frac{(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j))}{\Delta x}, \tag{2.3.15}$$

where the terms $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_1 := \frac{\Big(g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_j) V^n_j\Big)}{2}, \quad \mathcal{D}_2 := \frac{\Big(g(\rho^{(1)}_{j+\frac{1}{2},-}) V^{(1)}_{j+\frac{1}{2}} - g(\rho^{(1)}_j) V^{(1)}_j\Big)}{2}.$$

Next, with the observation $\rho_j^{(1)} = \frac{1}{2}(\rho^{(1)}_{j-\frac{1}{2},+} + \rho^{(1)}_{j+\frac{1}{2},-})$ and using the property (2.2.12) in conjunction with the Lemmas 2.2.3 and 2.2.6, we obtain a bound on the distance between $R^{(1)}_{j+\frac{1}{2}}$ and $R^{(1)}_j$ as

$$\begin{aligned}
&|R^{(1)}_{j+\frac{1}{2}} - R^{(1)}_j| \\
&= \frac{\Delta x}{2} \left| \sum_{k=0}^{N-1} w_\eta^k \left(\rho^{(1)}_{j+k+\frac{1}{2},+} - \frac{\rho^{(1)}_{j+k-\frac{1}{2},+}}{\hat{Q}_{\Delta x}} \right) + \sum_{k=0}^{N-1} w_\eta^{k+1} \left(\rho^{(1)}_{j+k+\frac{3}{2},-} - \frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}} \right) \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{k=0}^{N-1} \frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}} (w^{k+1}_\eta - w^k_\eta) \Bigg| \\
&= \frac{\Delta x}{2} \Bigg| \sum_{k=0}^{N-1} w^k_\eta \left(\rho^{(1)}_{j+k+\frac{1}{2},+} - \rho^{(1)}_{j+k-\frac{1}{2},+} + \rho^{(1)}_{j+k-\frac{1}{2},+} (\frac{\hat{Q}_{\Delta x} - 1}{\hat{Q}_{\Delta x}}) \right) \\
&\quad + \sum_{k=0}^{N-1} w^{k+1}_\eta \left(\rho^{(1)}_{j+k+\frac{3}{2},-} - \rho^{(1)}_{j+k+\frac{1}{2},-} + \rho^{(1)}_{j+k+\frac{1}{2},-} (\frac{\hat{Q}_{\Delta x} - 1}{\hat{Q}_{\Delta x}}) \right) \\
&\quad + \sum_{k=0}^{N-1} \frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}} (w^{k+1}_\eta - w^k_\eta) \Bigg| \\
&\leq \frac{\Delta x}{2} \left(w^0_\eta \sum_{k\in\mathbb{Z}} |\rho^{(1)}_{j+k+\frac{1}{2},+} - \rho^{(1)}_{j+k-\frac{1}{2},+}| + w^0_\eta \|\rho^{(1)}\| L \sum_{k=0}^{N-1} \Delta x \right. \\
&\quad \left. + w^0_\eta \sum_{k\in\mathbb{Z}} |\rho^{(1)}_{j+k+\frac{3}{2},-} - \rho^{(1)}_{j+k+\frac{1}{2},-}| + w^0_\eta \|\rho^{(1)}\| L \sum_{k=0}^{N-1} \Delta x + \|\rho^{(1)}\| \frac{w^0_\eta}{\hat{Q}_{\Delta x}} \right) \qquad (2.3.16) \\
&\leq \frac{\Delta x}{2} \left(2w^0_\eta \sum_{k\in\mathbb{Z}} |\rho^{(1)}_{j+k+1} - \rho^{(1)}_{j+k}| + 2w^0_\eta \|\rho^{(1)}\| \eta L + 2w^0_\eta \sum_{k\in\mathbb{Z}} |\rho^{(1)}_{j+k+1} - \rho^{(1)}_{j+k}| \right. \\
&\qquad \left. + \|\rho^{(1)}\| \frac{w^0_\eta}{\hat{Q}_{\Delta x}} \right) \\
&\leq \frac{\Delta x}{2} \left(4w^0_\eta (1 + C\Delta t) \mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta \rho_{\max} \eta L + \rho_{\max} \hat{w}^0_\eta \right).
\end{aligned}
$$

Subsequently, by subtracting and adding $g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_j$ to the term $\mathcal{D}_2$ and applying the estimate (2.3.16) as well as the property (2.2.12), we obtain

$$
\begin{aligned}
2|\mathcal{D}_2| &\leq \left| g\big(\rho^{(1)}_{j+\frac{1}{2},-}\big) \big(V^{(1)}_{j+\frac{1}{2}} - V^{(1)}_j \big) \right| + \left| \big(g(\rho^{(1)}_{j+\frac{1}{2},-}) - g(\rho^{(1)}_j) \big) V^{(1)}_j \right| \\
&\leq \|g\| \|v'\| |R^{(1)}_{j+\frac{1}{2}} - R^{(1)}_j| + \|g'\| \|v\| |\rho^{(1)}_{j+\frac{1}{2},-} - \rho^{(1)}_j| \\
&\leq \|g\| \|v'\| \frac{\Delta x}{2} \left(4w^0_\eta (1 + C\Delta t) \mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta \rho_{\max} \eta L + \rho_{\max} \hat{w}^0_\eta \right) \\
&\quad + \|g'\| \|v\| \frac{|\sigma^{(1)}_j|}{2} \\
&\leq \|g\| \|v'\| \frac{\Delta x}{2} \left(4w^0_\eta (1 + C\Delta t) \mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta \rho_{\max} \eta L + \rho_{\max} \hat{w}^0_\eta \right) \\
&\quad + \|g'\| \|v\| \frac{|\rho^{(1)}_{j+1} - \rho^{(1)}_j|}{2}.
\end{aligned}
$$

To proceed further, we observe the following inequality

$$\begin{aligned}
&\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}|\rho_{j+1}^{(1)}-\rho_j^{(1)}| \\
&\quad\leq \Delta t\Delta x(1+C\Delta t)\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}|\rho_{j+1}^{n}-\rho_j^{n}| \\
&\quad= (1+C\Delta t)\int_0^T\int_{-R}^{R}|\rho_{\Delta x}(t,x+\Delta x)-\rho_{\Delta x}(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq (1+C\Delta t)\int_0^T \Delta x\mathrm{TV}(\rho_{\Delta x}(t,\cdot))\,\mathrm{d}t \\
&\quad\leq \Delta x(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0)T.
\end{aligned} \tag{2.3.17}$$

Now, using (2.3.17) we obtain the bound

$$\begin{aligned}
&|\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\mathcal{D}_2\frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x}| \\
&\leq \Delta t\Delta x\|\varphi_x\|\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}\|g\|\|v'\|\frac{\Delta x}{4}\left(4w_\eta^0(1+C\Delta t)\mathrm{TV}(\rho_{\Delta x}^n)+2w_\eta^0\rho_{\max}\eta L+\rho_{\max}\hat{w}_\eta^0\right) \\
&\quad+\Delta t\Delta x\|\varphi_x\|\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}\|g'\|\|v\|\frac{|\rho_{j+1}^{(1)}-\rho_j^{(1)}|}{4} \\
&\leq \frac{\Delta x}{4}\|\varphi_x\|\|g\|\|v'\|\left(4w_\eta^0(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0)+2w_\eta^0\rho_{\max}\eta L+\rho_{\max}\hat{w}_\eta^0\right)2RT \\
&\quad+\frac{\Delta x}{4}\|\varphi_x\|\|g'\|\|v\|(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0)T.
\end{aligned} \tag{2.3.18}$$

In a similar way, a bound can be obtained on the term involving $\mathcal{D}_1$ in (2.3.15) as follows

$$\begin{aligned}
&\Big|\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\mathcal{D}_1\frac{\big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\big)}{\Delta x}\Big| \\
&\leq \frac{\Delta x}{4}\|\varphi_x\|\|g\|\|v'\|\left(4w_\eta^0\exp(2TC)\mathrm{TV}(\rho_0)+2w_\eta^0\rho_{\max}\eta L+\rho_{\max}\hat{w}_\eta^0\right)2RT \\
&\quad+\frac{\Delta x}{4}\|\varphi_x\|\|g'\|\|v\|\exp(2TC)\mathrm{TV}(\rho_0)T.
\end{aligned} \tag{2.3.19}$$

Combining the estimates (2.3.18) and (2.3.19), we see that

$$\lim_{\Delta x\to 0}\mathcal{S}_1=0. \tag{2.3.20}$$

Finally, collecting the results (2.3.8), (2.3.14) and (2.3.20), we can conclude that

$$\begin{aligned}
0=\lim_{\Delta x\to 0}(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)&=\lim_{\Delta x\to 0}(\mathcal{P}_1+\mathcal{P}_2)+\lim_{\Delta x\to 0}(\mathcal{S}_1+\mathcal{S}_2) \\
&=\int_{-\infty}^{+\infty}\rho(0,x)\varphi(0,x)dx+\int_0^T\int_{-\infty}^{+\infty}\rho(t,x)\varphi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$+\int_0^T \int_{-\infty}^{+\infty} g\big(\rho(t,x)\big) v\big(\rho * w_\eta(t,x)\big) \varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

This reveals that the limit is a weak solution of the problem (2.1.1). □

Remark 2.3.3. It is important to note that in the case when $g(\rho) = \rho$, the weak solutions are unique and no entropy condition is required, as discussed in [119]. Hence, in this specific case, we can conclude that the second-order scheme (2.2.8) converges to the unique weak solution without any entropy condition. However, for the general case, it is required to prove the convergence to the entropy solution. This will be discussed in the next section.

2.4 Convergence to the entropy solution

To prove convergence to the entropy solution, we shall use the same approach as outlined in [167], also see [166]. These ideas can be combined to form the following theorem which is analogous to the Theorem 3.1 of [166].

Theorem 2.4.1. *Suppose that a scheme can be written in the form:*

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1} - a_{j+\frac{1}{2}}^{n+1} + a_{j-\frac{1}{2}}^{n+1}, \tag{2.4.1}$$

where
(i) $\tilde{\rho}_j^{n+1}$ *is computed from* ρ_j^n*, using a scheme which yields a sequence of approximate solutions converging in* $\mathrm{L}^1_{\mathrm{loc}}$ *to the entropy solution of* (2.1.1).
(ii) $|a_{j+\frac{1}{2}}^{n+1}| \leq K\Delta x^\delta$ *for some constant* K *which is independent of* Δx *and for some* $\delta \in (0,1)$.
(iii) The approximate solutions $\rho_{\Delta x}$ *obtained using* (2.4.1) *are in* BV, L^∞ *and admits* L^1-*Lipschitz continuity in time.*
Then the approximate solutions generated by the scheme (2.4.1) converges in $\mathrm{L}^1_{\mathrm{loc}}$ *to the entropy solution of* (2.1.1).

Remark 2.4.2. Proof of Theorem 2.4.1 follows along the same lines as that of Theorem 3.1 of [166]. Specifically, the hypothesis (iii) of Theorem 2.4.1 ensures that the approximate solutions generated by the scheme (2.4.1) converges in $\mathrm{L}^1_{\mathrm{loc}}$. To prove that the limit solution satisfies the entropy condition (2.1.4), we mainly use two facts. Firstly, we utilize the discrete entropy inequality of the scheme $\tilde{\rho}_j^{n+1}$ in hypothesis (i) of Theorem 2.4.1, which is provided later in equation (2.4.5) of Theorem 2.4.4. Secondly, we make use of the boundedness of the terms $|a_{j+\frac{1}{2}}^{n+1}| \leq K\Delta x^\delta$ as mentioned in hypothesis (ii) of Theorem 2.4.1, and the BV and L^∞ estimates of ρ_j^{n+1}. By adding an appropriate term to both sides of the discrete entropy inequality (2.4.5) of Theorem (2.4.4), we get a similar expression as in the inequality (3.18) of [166]. Further, by employing a similar argument as presented in [166], one can show that the right-hand side of the obtained expression has a limit supremum that is non-positive as the mesh size approaches zero. Consequently, the required result follows.

In this scenario, first we consider the first-order in space and second-order in time scheme (FSST) obtained by setting the slopes $\sigma_j(t) = 0$ for all $j \in \mathbb{Z}$ in (2.2.8):

$$\begin{aligned}
\rho_j^{(1)} &= \rho_j^n - \lambda \left(g(\rho_j^n) V_{j+\frac{1}{2}}^n - g(\rho_{j-1}^n) V_{j-\frac{1}{2}}^n \right), \\
\rho_j^{(2)} &= \rho_j^{(1)} - \lambda \left(g(\rho_j^{(1)}) V_{j+\frac{1}{2}}^{(1)} - g(\rho_{j-1}^{(1)}) V_{j-\frac{1}{2}}^{(1)} \right), \\
\rho_j^{n+1} &= \frac{1}{2} \left(\rho_j^n + \rho_j^{(2)} \right),
\end{aligned} \tag{2.4.2}$$

where

$$\begin{aligned}
&V_{j+\frac{1}{2}}^n = v(R_{j+\frac{1}{2}}^n), \quad V_{j+\frac{1}{2}}^{(1)} = v(R_{j+\frac{1}{2}}^{(1)}), \\
&R_{j+\frac{1}{2}}^n = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^n \frac{(w_\eta^k + w_\eta^{k+1})}{2} \text{ and } R_{j+\frac{1}{2}}^{(1)} = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^{(1)} \frac{(w_\eta^k + w_\eta^{k+1})}{2}.
\end{aligned}$$

Further, for some $K > 0$ and $\delta \in (0,1)$, we modify the slopes $\sigma_j(t)$ defined in (2.2.4) by adding the term $K\Delta x^\delta$ in its definition, i.e.,

$$\sigma_j(t) = \mathrm{mm}((\rho_j(t) - \rho_{j-1}(t)), \ \frac{(\rho_{j+1}(t) - \rho_{j-1}(t))}{2}, (\rho_{j+1}(t) - \rho_j(t)), \ s_{j-\frac{1}{2}}(t) K \Delta x^\delta). \tag{2.4.3}$$

where mm denotes the minmod function (2.2.5) and $s_{j-\frac{1}{2}}(t) := \operatorname{sgn}(\rho_j(t) - \rho_{j-1}(t))$. Now, the second-order scheme (2.2.8) with the modified slope (2.4.3) can be written as a predictor-corrector scheme in the form

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1} - a_{j+\frac{1}{2}}^{n+1} + a_{j-\frac{1}{2}}^{n+1}, \tag{2.4.4}$$

where $a_{j+\frac{1}{2}}^{n+1} = \lambda(F_{j+\frac{1}{2}}^n - \tilde{F}_{j+\frac{1}{2}}^n)$ with $F_{j+\frac{1}{2}}^n$ as in (2.2.9) and $\tilde{\rho}_j^{n+1}$ is a predictor step obtained from ρ_j^n using the FSST scheme (2.4.2), written as

$$\tilde{\rho}_j^{n+1} = \rho_j^n - \lambda(\tilde{F}_{j+\frac{1}{2}}^n - \tilde{F}_{j-\frac{1}{2}}^n), \ \text{ where } \tilde{F}_{j+\frac{1}{2}}^n = \frac{1}{2} \left(g(\rho_j^n) \tilde{V}_{j+\frac{1}{2}}^n + g(\tilde{\rho}_j^{(1)}) \tilde{V}_{j+\frac{1}{2}}^{(1)} \right)$$

and

$$\begin{aligned}
&\tilde{\rho}_j^{(1)} = \rho_j^n - \lambda \left(g(\rho_j^n) \tilde{V}_{j+\frac{1}{2}}^n - g(\rho_{j-1}^n) \tilde{V}_{j-\frac{1}{2}}^n \right), \ \tilde{V}_{j+\frac{1}{2}}^n = v(\tilde{R}_{j+\frac{1}{2}}^n), \ \ \tilde{V}_{j+\frac{1}{2}}^{(1)} = v(\tilde{R}_{j+\frac{1}{2}}^{(1)}) \\
&\tilde{R}_{j+\frac{1}{2}}^n = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^n \frac{(w_\eta^k + w_\eta^{k+1})}{2}, \ \tilde{R}_{j+\frac{1}{2}}^{(1)} = \Delta x \sum_{k=0}^{N-1} \tilde{\rho}_{j+k+1}^{(1)} \frac{(w_\eta^k + w_\eta^{k+1})}{2} \quad \text{for all } j \in \mathbb{Z}.
\end{aligned}$$

We now state our final result in the following theorem which ensures convergence to the entropy solution and it will be proved using Theorem 2.4.1.

Theorem 2.4.3. *(Convergence to the entropy solution) Let* $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ *and let* $\rho_{\Delta x}$ *be the approximate solution obtained using the second-order scheme* (2.2.8) *under the CFL condition* (2.2.13)*, with a space-step dependent slope limiter* (2.4.3)*. Then, the corresponding sequence of approximate solutions* $\rho_{\Delta x}$ *converges in* $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$ *to the unique entropy solution of (2.1.1) as* $\Delta x \to 0$*.*

As the first step in proving Theorem 2.4.3, we show that the numerical solutions obtained by the scheme (2.4.2) converges to the entropy solution of (2.1.1).

Theorem 2.4.4. *Let* $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ *and let* $\rho_{\Delta x}$ *be the approximate solution obtained using the FSST scheme* (2.4.2) *under the CFL condition* (2.2.13)*. Then* $\rho_{\Delta x}$ *converges in* $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$ *to the unique entropy solution of (2.1.1) as* $\Delta x \to 0$*.*

Proof. It is clear that the convergence analysis (Theorem 2.3.2) presented in Section 2.3 holds for the FSST scheme by setting $\sigma_j(t)=0$ for all j in the scheme (2.2.8). Therefore, it is only left to prove that the limit function ρ obtained from the FSST scheme (2.4.2) satisfies the entropy condition (2.1.4). To prove this, first we observe that the first-order time steps in the scheme (2.4.2) satisfy the following discrete entropy inequalities (see [89]):

$$\begin{aligned}
|\rho_j^{(1)}-\kappa| - |\rho_j^n-\kappa| + \lambda\left(F^{\kappa}_{j+\frac12}(\rho_j^n) - F^{\kappa}_{j-\frac12}(\rho_{j-1}^n)\right)&\\
+\lambda g(\kappa)\mathrm{sgn}(\rho_j^{(1)}-\kappa)(V^n_{j+\frac12}-V^n_{j-\frac12}) &\le 0,\\
|\rho_j^{(2)}-\kappa| - |\rho_j^{(1)}-\kappa| + \lambda\left(F^{\kappa,(1)}_{j+\frac12}(\rho_j^{(1)}) - F^{\kappa,(1)}_{j-\frac12}(\rho_{j-1}^{(1)})\right)&\\
+\lambda g(\kappa)\mathrm{sgn}(\rho_j^{(2)}-\kappa)(V^{(1)}_{j+\frac12}-V^{(1)}_{j-\frac12}) &\le 0,
\end{aligned}$$

where $\kappa \in I=[0,\rho_{\max}]$, $F^{\kappa}_{j+\frac12}(\rho) := \Big(g(\rho\wedge\kappa)-g(\rho\vee\kappa)\Big)V^n_{j+\frac12}$, $F^{\kappa,(1)}_{j+\frac12}(\rho) := \Big(g(\rho\wedge\kappa)-g(\rho\vee\kappa)\Big)V^{(1)}_{j+\frac12}$, $a\wedge b := \max\{a,b\}$ and $a\vee b := \min\{a,b\}$. Combining these, we obtain a discrete entropy inequality for the FSST scheme (2.4.2) as follows

$$\begin{aligned}
|\rho_j^{n+1}-\kappa| - |\rho_j^n-\kappa| + \frac{\lambda}{2}\left[F^{\kappa,(1)}_{j+\frac12}(\rho_j^{(1)}) + F^{\kappa}_{j+\frac12}(\rho_j^n) - F^{\kappa,(1)}_{j-\frac12}(\rho_{j-1}^{(1)}) - F^{\kappa}_{j-\frac12}(\rho_{j-1}^n)\right]&\\
+\frac{\lambda}{2}g(\kappa)\mathrm{sgn}(\rho_j^{(2)}-\kappa)(V^{(1)}_{j+\frac12}-V^{(1)}_{j-\frac12}) + \frac{\lambda}{2}g(\kappa)\mathrm{sgn}(\rho_j^{(1)}-\kappa)(V^n_{j+\frac12}-V^n_{j-\frac12}) &\le 0.
\end{aligned} \tag{2.4.5}$$

From (2.4.5), we prove that the approximate solutions converge to the entropy solution as in (2.1.4) of Definition 2.1.2.

Now, consider a non-negative test function $\varphi \in \mathrm{C}^1_c([0,T)\times\mathbb{R};\mathbb{R}_+)$. Let T_φ be such that $0\le T_\varphi < T$ and $\varphi(t,x)=0$ for $t\ge T_\varphi$ and let n_T be such that $T_\varphi \in (n_T\Delta t,(n_T+1)\Delta t]$. Multiplying (2.4.5) by $\Delta x\varphi(t^n,x_j)$, summing over n,j and using summation by parts, we obtain

$$\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\mathcal{E}_4 \ge 0, \tag{2.4.6}$$

where

$$\mathcal{E}_0 := \Delta x \sum_{j\in\mathbb{Z}} \varphi(0,x_j)|\rho_j^0 - \kappa| + \Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} |\rho_j^{n+1} - \kappa| \frac{\big(\varphi(t^{n+1},x_j) - \varphi(t^n,x_j)\big)}{\Delta t},$$

$$\mathcal{E}_1 := \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} F^{\kappa,(1)}_{j+\frac{1}{2}}(\rho_j^{(1)}) \frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$

$$\mathcal{E}_2 := \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} F^{\kappa,n}_{j+\frac{1}{2}}(\rho_j^n) \frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$

$$\mathcal{E}_3 := -\frac{\Delta t \Delta x}{2} g(\kappa) \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{(1)} - \kappa) \frac{(V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}})}{\Delta x} \varphi(t^n,x_j),$$

$$\mathcal{E}_4 := -\frac{\Delta t \Delta x}{2} g(\kappa) \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_j^{(2)} - \kappa) \frac{(V^{(1)}_{j+\frac{1}{2}} - V^{(1)}_{j-\frac{1}{2}})}{\Delta x} \varphi(t^n,x_j).$$

First, consider the term $\mathcal{E}_0$ in (2.4.6) which can be written as

$$\mathcal{E}_0 = \int_{-\infty}^{+\infty} \varphi_{\Delta x}(0,x)|\rho_{\Delta x}(0,x) - \kappa| \,\mathrm{d}x + \int_0^T \int_{-\infty}^{+\infty} |\rho_{\Delta x}(t+\Delta t,x) - \kappa| \partial_t \varphi_{\Delta x}(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$

where $\partial_t \varphi_{\Delta x}(t,x) := \varphi_t(\bar{t},x_j) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n,t^{n+1}), \text{ for some } \bar{t} \in (t^n,t^{n+1})$. Through the dominated convergence theorem it follows that

$$\lim_{\Delta x\to 0} \mathcal{E}_0 = \int_{-\infty}^{\infty} |\rho_0(x) - \kappa| \varphi(0,x) \,\mathrm{d}x + \int_0^T \int_{-\infty}^{\infty} |\rho(t,x) - \kappa| \partial_t \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t. \tag{2.4.7}$$

Let $R > 0$ be such that $\varphi(t,x) = 0$ for $|x| > R$. Let $j_0, j_1 \in \mathbb{Z}$ such that $-R \in (x_{j_0-\frac{1}{2}}, x_{j_0+\frac{1}{2}}]$ and $R \in (x_{j_1-\frac{1}{2}}, x_{j_1+\frac{1}{2}}]$. Next, we consider the term $\mathcal{E}_1$ in (2.4.6) which writes as

$$\mathcal{E}_1 = \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \Big(g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) \Big) V^{(1)}_{j+\frac{1}{2}} \frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x}. \tag{2.4.8}$$

Using the definition of $\rho^{(1)}$ and applying the mean value theorem, it follows that

$$V^{(1)}_{j+\frac{1}{2}} = V^n_{j+\frac{1}{2}} - \Delta t v'(\theta_{j+\frac{1}{2}}) \sum_{k=0}^{N-1} \frac{(w_\eta^k + w_\eta^{k+1})}{2} \Big(g(\rho^n_{j+k+1}) V^n_{j+k+\frac{3}{2}} - g(\rho^n_{j+k}) V^n_{j+k+\frac{1}{2}} \Big),$$

for some $\theta_{j+\frac{1}{2}} \in \mathcal{I}(R^n_{j+\frac{1}{2}}, R^{(1)}_{j+\frac{1}{2}})$. Thus the term $\mathcal{E}_1$ can be written as $\mathcal{E}_1 = \mathcal{E}_1^a + \mathcal{E}_1^b$ where

$$\mathcal{E}_1^a := \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \hat{q}(n,j) V^n_{j+\frac{1}{2}} \frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$

$$\mathcal{E}_1^b := -\frac{\Delta t^2 \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \hat{q}(n,j) v'(\theta_{j+\frac{1}{2}}) \sum_{k=0}^{N-1} \hat{\ell}(n,j,k) \frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$

with the definition, $\hat{\ell}(n,j,k) := \dfrac{(w_\eta^k + w_\eta^{k+1})}{2}\Big(g(\rho_{j+k+1}^n)V_{j+k+\frac{3}{2}}^n - g(\rho_{j+k}^n)V_{j+k+\frac{1}{2}}^n\Big)$ and $\hat{q}(n,j) := g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa)$. Now, summation by parts yields

$$\begin{aligned}\sum_{k=0}^{N-1}\hat{\ell}(n,j,k) = {} & \frac{w_\eta^{N-1}+w_\eta^N}{2}g(\rho_{j+N}^n)V_{j+N+\frac{1}{2}}^n - \frac{w_\eta^0+w_\eta^1}{2}g(\rho_j^n)V_{j+\frac{1}{2}}^n \\ & + \sum_{k=1}^{N-1} g(\rho_{j+k}^n)V_{j+k+\frac{1}{2}}^n\frac{(w_\eta^{k-1}-w_\eta^{k+1})}{2},\end{aligned}$$

which implies that

$$\begin{aligned}\left|\sum_{k=0}^{N-1}\hat{\ell}(n,j,k)\right| & \le 2w_\eta^0\|g\|\|v\| + \|g\|\|v\|\sum_{k=1}^{N-1}\frac{(w_\eta^{k-1}-w_\eta^{k+1})}{2} \\ & \le 2w_\eta^0\|g\|\|v\| + \|g\|\|v\|w_\eta^0 \le 3w_\eta^0\|g\|\|v\|.\end{aligned}$$

Therefore we have the following bound on $|\mathcal{E}_1^b|$:

$$|\mathcal{E}_1^b| \le 3\Delta t\|\varphi_x\|\|g\|^2\|v\|\|v'\|w_\eta^0\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\Delta t\Delta x \le 6\Delta t\|\varphi_x\|\|g\|^2\|v\|\|v'\|w_\eta^0 RT. \tag{2.4.9}$$

Since g is an increasing function, $g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) = |g(\rho_j^{(1)}) - g(\kappa)|$. Now, note that $\mathcal{E}_1^a$ can be written as follows

$$\mathcal{E}_1^a = \frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\Big(\big|g\big(\rho_{\Delta x}^{(1)}(t,x)\big) - g(\kappa)\big|v\big(R_{\Delta x}(t,x+\Delta x)\big)\Big)\,\partial_x\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\begin{aligned} R_{\Delta x}(t,x) &:= R_{j-\frac{1}{2}}^n \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}), \\ \partial_x\varphi_{\Delta x}(t,x) &:= \varphi_x(t^n,\bar{x}) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}],\ t \in [t^n, t^{n+1}) \text{ for some } \bar{x} \in (x_j, x_{j+1}).\end{aligned}$$

Observe that

$$\begin{aligned} & R_{\Delta x}(t,x+\Delta x) \\ & = \frac{1}{2}\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta}\rho_{\Delta x}(t,y+\Delta x)\Big(w_{\eta,\Delta x}(y-x_{j-\frac{1}{2}}+\Delta x) + w_{\eta,\Delta x}(y-x_{j-\frac{1}{2}}+2\Delta x)\Big)\,\mathrm{d}y\end{aligned}$$

for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, where

$$w_{\eta,\Delta x}(x) := w_\eta^k \quad \text{for } x \in (k\Delta x, (k+1)\Delta x], \quad w_{\eta,\Delta x}(0) := w_\eta(0).$$

By using Lemma A.3.1 (see Appendix A.3) and applying the dominated convergence theorem, it follows that $R_{\Delta x}(t,x+\Delta x)$ converges to $\int_x^{x+\eta}\rho(t,y)w_\eta(y-x)\,\mathrm{d}y$ as $\Delta x \to 0$. Using Lemma A.3.2 (see Appendix A.3) and the dominated convergence theorem, we deduce that

$$\lim_{\Delta x\to 0}\mathcal{E}_1^a = \frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\big(\big|g\big(\rho(t,x)\big) - g(\kappa)\big|v\big(\rho * w_\eta(t,x)\big)\big)\,\varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.4.10}$$

The term $\mathcal{E}_2$ in (2.4.6) can be expressed as follows

$$\begin{aligned}\mathcal{E}_2 &= \frac{\Delta t \Delta x}{2}\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\left(g(\rho_j^n\wedge\kappa)-g(\rho_j^n\vee\kappa)\right)V_{j+\frac{1}{2}}^n\frac{\left(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\right)}{\Delta x}\\ &= \frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\Big(\big|g\big(\rho_{\Delta x}(t,x)\big)-g(\kappa)\big|v\big(R_{\Delta x}(t,x+\Delta x)\big)\Big)\partial_x\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using the convergence of $\rho_{\Delta x}$ to ρ and similar arguments as in the case of $\mathcal{E}_1^a$, we observe that

$$\lim_{\Delta x\to 0}\mathcal{E}_2 = \frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\left(\big|g\big(\rho(t,x)\big)-g(\kappa)\big|v\big(\rho * w_\eta(t,x)\big)\right)\varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.4.11}$$

Further, the term $\mathcal{E}_3$ in (2.4.6) can be written as

$$\begin{aligned}\mathcal{E}_3 &= -\frac{\Delta t\Delta x}{2}g(\kappa)\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\operatorname{sgn}\big(\rho_j^{(1)}-\kappa\big)v'(\bar{R}_j^n)\frac{(R_{j+\frac{1}{2}}^n-R_{j-\frac{1}{2}}^n)}{\Delta x}\varphi(t^n,x_j)\\ &= -\frac{1}{2}g(\kappa)\int_0^T\int_{-\infty}^{+\infty}\Lambda(t,x)\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\Lambda(t,x) := \operatorname{sgn}\left(\rho_{\Delta x}^{(1)}(t,x)-\kappa\right)v'\big(\bar{R}_{\Delta x}(t,x)\big)\frac{\left(R_{\Delta x}(t,x+\Delta x)-R_{\Delta x}(t,x)\right)}{\Delta x}$, $\bar{R}_j^n \in \mathcal{I}(R_{j-\frac{1}{2}}^n, R_{j+\frac{1}{2}}^n)$, $\bar{R}_{\Delta x}(t,x) := \bar{R}_j^n$ for $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$, and $\varphi_{\Delta x} := \varphi(t^n,x_j)$ for $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$.
Using summation by parts, we obtain

$$\frac{R_{j+\frac{1}{2}}^n-R_{j-\frac{1}{2}}^n}{\Delta x} = \frac{w_\eta^{N-1}+w_\eta^N}{2}\rho_{j+N}^n - \frac{w_\eta^0+w_\eta^1}{2}\rho_j^n - \Delta x\sum_{k=1}^{N-1}\rho_{j+k}^n\frac{w_\eta^{k+1}-w_\eta^{k-1}}{2\Delta x}, \tag{2.4.12}$$

which enables us to write

$$\begin{aligned}\frac{R_{\Delta x}(t,x+\Delta x)-R_{\Delta x}(t,x)}{\Delta x} &= \rho_{\Delta x}(t^n,x_{j+N})\left(\frac{w_{\eta,\Delta x}(\eta-\Delta x)+w_{\eta,\Delta x}(\eta)}{2}\right)\\ &\quad - \rho_{\Delta x}(t^n,x_j)\left(\frac{w_{\eta,\Delta x}(0)+w_{\eta,\Delta x}(\Delta x)}{2}\right)\\ &\quad - \int_{x_{j-\frac{1}{2}}+\Delta x}^{x_{j-\frac{1}{2}}+\eta}\rho_{\Delta x}(t,y)w'_{\eta,\Delta x}\big(y-(x_{j-\frac{1}{2}}+\Delta x)\big)\,\mathrm{d}y,\end{aligned} \tag{2.4.13}$$

for $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$, where $w'_{\eta,\Delta x}(x) := w'_\eta(\bar{x}),\ x\in(k\Delta x,(k+1)\Delta x]$ for some $\bar{x}\in(k\Delta x,(k+2)\Delta x)$. Defining $R(t,x) := \int_x^{x+\eta}\rho(t,y)w_\eta(y-x)\,\mathrm{d}y$ and differentiating yields

$$\frac{\partial R(t,x)}{\partial x} = \rho(t,x+\eta)w_\eta(\eta)-\rho(t,x)w_\eta(0)-\int_x^{x+\eta}\rho(t,y)w'_\eta(y-x)\,\mathrm{d}y. \tag{2.4.14}$$

Using the dominated convergence theorem in (2.4.13), we have the following for a.e. $(t,x)\in[0,T)\times\mathbb{R}$,

$$\lim_{\Delta x\to 0}\frac{\big(R_{\Delta x}(t,x+\Delta x)-R_{\Delta x}(t,x)\big)}{\Delta x} = \frac{\partial R(t,x)}{\partial x}.$$

Now, using Lemma A.3.2 (see Appendix A.3) together with the arguments in Lemma 4.3 and Lemma 4.4 of [115], the following holds for $\kappa \in I = [0, \rho_{\max}]$

$$\begin{aligned}\lim_{\Delta x \to 0} &\mathcal{E}_3 \\ &= -\frac{1}{2}\int_0^T \int_{-\infty}^{+\infty} \operatorname{sgn}\big(\rho(t,x)-\kappa\big) g(\kappa) v'\left(\rho * w_\eta(t,x)\right) \partial_x\big(\rho * w_\eta(t,x)\big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.4.15}$$

Further, we consider the term $\mathcal{E}_4$ in (2.4.6) which can be expressed as follows

$$\mathcal{E}_4 = -\frac{1}{2} g(\kappa) \int_0^T \int_{-\infty}^{+\infty} \hat{\lambda}(t,x)\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where $\hat{\lambda}(t,x) := \operatorname{sgn}\big(\rho^{(2)}_{\Delta x}(t,x)-\kappa\big) v'\big(\bar{R}^{(1)}_{\Delta x}(t,x)\big) \dfrac{\big(R^{(1)}_{\Delta x}(t,x+\Delta x) - R^{(1)}_{\Delta x}(t,x)\big)}{\Delta x}$, $\bar{R}^{(1)}_{\Delta x}(t,x) := \bar{R}^{(1)}_j$, $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$ for some $\bar{R}^{(1)}_j \in \mathcal{I}(R^{(1)}_{j-\frac{1}{2}}, R^{(1)}_{j+\frac{1}{2}})$ and $R^{(1)}_{\Delta x}(t,x) := R^{(1)}_{j-\frac{1}{2}}$ for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$. Proceeding in a way similar to the derivation of (2.4.15), we get

$$\begin{aligned}\lim_{\Delta x \to 0} &\mathcal{E}_4 \\ &= -\frac{1}{2}\int_0^T \int_{-\infty}^{+\infty} \operatorname{sgn}\big(\rho(t,x)-\kappa\big) g(\kappa) v'\big(\rho * w_\eta(t,x)\big) \partial_x\big(\rho * w_\eta(t,x)\big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.4.16}$$

Finally, collecting the expressions (2.4.7), (2.4.9), (2.4.10), (2.4.11), (2.4.15) and (2.4.16), we obtain the desired entropy inequality

$$\begin{aligned}&\int_0^T \int_{-\infty}^{+\infty} \Big(|\rho-\kappa|\partial_t\varphi + \operatorname{sgn}(\rho-\kappa)\big(g(\rho)-g(\kappa)\big)v(\rho * w_\eta)\partial_x\varphi \\ &- \operatorname{sgn}(\rho-\kappa) g(\kappa) v'(\rho * w_\eta)\partial_x(\rho * w_\eta)\varphi\Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} |\rho_0(x)-\kappa|\varphi(0,x)\,\mathrm{d}x \geq 0.\end{aligned}$$

□

In the following lemma we show that the bound in condition (ii) of Theorem 2.4.1 holds for the terms $a^{n+1}_{j+\frac{1}{2}}$ in (2.4.4).

Lemma 2.4.5. *Consider the second-order scheme with the modified slope* (2.4.3) *written in the form* (2.4.4). *Then* $|a^{n+1}_{j+\frac{1}{2}}| \leq \tilde{K}\Delta x^\delta$ *for some constant* $\tilde{K}$ *which is independent of* Δx *and* $\delta \in (0,1)$ *as in* (2.4.3).

Proof. We can write

$$\begin{aligned}a^{n+1}_{j+\frac{1}{2}} &= \frac{\lambda}{2}\Big(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} + g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}} - g(\rho^n_j)\tilde{V}^n_{j+\frac{1}{2}} - g(\tilde{\rho}^{(1)}_j)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big) \\ &= \frac{\lambda}{2}\Big(\big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_j)\big)V^n_{j+\frac{1}{2}} + g(\rho^n_j)(V^n_{j+\frac{1}{2}} - \tilde{V}^n_{j+\frac{1}{2}})\end{aligned}$$

$$
\begin{aligned}
&+\Big(g(\rho^{(1)}_{j+\frac{1}{2},-}) - g(\rho^{(1)}_j)\Big)V^{(1)}_{j+\frac{1}{2}} + g(\rho^{(1)}_j)(V^{(1)}_{j+\frac{1}{2}} - \tilde{V}^{(1)}_{j+\frac{1}{2}}) + \Big(g(\rho^{(1)}_j) - g(\tilde{\rho}^{(1)}_j)\Big)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big) \\
&= \frac{\lambda}{2}\Big(g'(\xi_j)(\rho^n_{j+\frac{1}{2},-} - \rho^n_j)V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\eta_{j+\frac{1}{2}})(R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}}) \\
&+ g'(\xi^{(1)}_j)(\rho^{(1)}_{j+\frac{1}{2},-} - \rho^{(1)}_j)V^{(1)}_{j+\frac{1}{2}} + g(\rho^n_j)v'(\eta^{(1)}_{j+\frac{1}{2}})(R^{(1)}_{j+\frac{1}{2}} - \tilde{R}^{(1)}_{j+\frac{1}{2}}) \\
&+g'(\bar{\xi}^{(1)}_j)(\rho^{(1)}_j - \tilde{\rho}^{(1)}_j)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big),
\end{aligned}
$$

for some suitable $\xi_j \in \mathcal{I}(\rho^n_{j+\frac{1}{2},-}, \rho^n_j), \eta_{j+\frac{1}{2}} \in \mathcal{I}(R^n_{j+\frac{1}{2}}, \tilde{R}^n_{j+\frac{1}{2}}), \xi^{(1)}_j \in \mathcal{I}(\rho^{(1)}_{j+\frac{1}{2},-}, \rho^{(1)}_j), \bar{\xi}^{(1)}_j \in \mathcal{I}(\rho^{(1)}_j, \tilde{\rho}^{(1)}_j)$ and $\eta^{(1)}_{j+\frac{1}{2}} \in \mathcal{I}(R^{(1)}_{j+\frac{1}{2}}, \tilde{R}^{(1)}_{j+\frac{1}{2}})$ by the mean value theorem. Also, note that the term $\rho^{(1)}_j - \tilde{\rho}^{(1)}_j$ can be written as

$$
\begin{aligned}
\rho^{(1)}_j - \tilde{\rho}^{(1)}_j = -\lambda\Big(&g'(\zeta_j)(\rho^n_{j+\frac{1}{2},-} - \rho^n_j)V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\theta_{j+\frac{1}{2}})(R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}}) \\
&+ g'(\zeta_{j-1})(\rho^n_{j-1} - \rho^n_{j-\frac{1}{2},-})V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\theta_{j-\frac{1}{2}})(\tilde{R}^n_{j-\frac{1}{2}} - R^n_{j-\frac{1}{2}})\Big),
\end{aligned}
$$

for some $\zeta_j \in \mathcal{I}(\rho^n_{j+\frac{1}{2},-}, \rho^n_j), \theta_{j+\frac{1}{2}} \in \mathcal{I}(R^n_{j+\frac{1}{2}}, \tilde{R}^n_{j+\frac{1}{2}})$. By the definition of slopes (2.4.3) and using property (2.2.2), we can easily see that

$$
\begin{aligned}
&|\rho^n_{j+\frac{1}{2}-} - \rho^n_j|, \ \ |\rho^{(1)}_{j+\frac{1}{2}-} - \rho^{(1)}_j| \le \frac{1}{2}K\Delta x^\delta, \\
&|R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}}| \le K_1\Delta x^\delta, \ \ |R^{(1)}_{j+\frac{1}{2}} - \tilde{R}^{(1)}_{j+\frac{1}{2}}| \le K_2\Delta x^\delta, \ \ |\rho^{(1)}_j - \tilde{\rho}^{(1)}_j| \le K_3\Delta x^\delta,
\end{aligned}
$$

where $K_1 := \frac{1}{2}Kw^0_\eta\eta,\ K_2 := w^0_\eta\eta\Big(\lambda K\big(\|g'\|\|v\|+\|g\|\|v'\|w^0_\eta\eta\big)+\frac{K}{2}\Big)$ and $K_3 := \lambda K\big(\|g'\|\|v\|+ \|g\|\|v'\|w^0_\eta\eta\big)$. Now, defining $K_4 := \max\big\{\frac{K}{2}, K_1, K_2, K_3\big\}$ and $\tilde{K} := \frac{\lambda}{2}K_4\big(3\|g'\|\|v\| + 2\|g\|\|v'\|\big)$, we can conclude that

$$
|a^{n+1}_{j+\frac{1}{2}}| \le \tilde{K}\Delta x^\delta, \ \ \delta \in (0,1).
$$

This completes the proof. □

Proof of Theorem 2.4.3: The second-order scheme (2.2.8) with the modified slope (2.4.3) can be written in the form (2.4.1). Further, the hypotheses (i) and (ii) of the Theorem 2.4.1 are satisfied through Theorem 2.4.4 and Lemma 2.4.5, respectively. Theorems 2.2.4, 2.2.7, and 2.2.9 hold for the scheme (2.2.8) even with the modified slopes (2.4.3), thereby proving hypothesis (iii) of Theorem 2.4.1. Thus, using Theorem 2.4.1 we can conclude that with the modified slope (2.4.3), the second-order scheme (2.2.8) converges to the unique entropy solution of (2.1.1).

Remark 2.4.6. In fact, the modification in the slope is needed only for the analysis, in implementation it is not needed (see [131, 166, 167, 165, 105]). Specifically, it is mentioned just below equation (26), page number 158 of [131] and just below Figure 3, page 68 of [166]. Also see the Remark in page 577 of [165].

2.5 A MUSCL-Hancock type scheme

In this section, we propose a MUSCL- Hancock type second-order accurate method for approximating the problem given in (2.1.1). We discretize the domain with parameters Δx and Δt as in Section 2.2. Given the cell average values ρ_j^n at time $t = t^n$, we reconstruct a piecewise linear function denoted by $\tilde{\rho}^n$ as

$$\tilde{\rho}^n(x) = \rho_j^n + \frac{(x - x_j)}{\Delta x}\sigma_j^n \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

where the slope σ_j^n is chosen as in (2.2.4). To compute the solution at the next time level t^{n+1}, we follow two steps.

Step 1: The left and right face values at each interface are evolved in time by a unit of $\frac{\Delta t}{2}$ using the Taylor expansion:

$$\begin{aligned}
\rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}} &= \rho_{j+\frac{1}{2},-}^{n} - \frac{\lambda}{2}\left(g(\rho_{j+\frac{1}{2},-}^{n})v(R_{j+\frac{1}{2},-}^{n}) - g(\rho_{j-\frac{1}{2},+}^{n})v(R_{j-\frac{1}{2},+}^{n})\right), \\
\rho_{j+\frac{1}{2},+}^{n+\frac{1}{2}} &= \rho_{j+\frac{1}{2},+}^{n} - \frac{\lambda}{2}\left(g(\rho_{j+\frac{3}{2},-}^{n})v(R_{j+\frac{3}{2},-}^{n}) - g(\rho_{j+\frac{1}{2},+}^{n})v(R_{j+\frac{1}{2},+}^{n})\right),
\end{aligned} \tag{2.5.1}$$

where the convolution terms $R_{j+\frac{1}{2},\pm}^n$ are computed as

$$R_{j+\frac{1}{2},-}^n = \Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+\frac{1}{2},-}^n \quad \text{and} \quad R_{j+\frac{1}{2},+}^n = \Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+\frac{1}{2},+}^n.$$

Step 2: The updated approximate solution at time t^{n+1} is given by

$$\rho_j^{n+1} = \rho_j^n - \lambda(f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n),$$

where $f_{j+\frac{1}{2}}^n$ is the numerical flux. Here, we use a Godunov-type numerical flux (as given in [89]) defined by

$$f_{j+\frac{1}{2}}^n = g(\rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}})v(R_{j+\frac{1}{2}}^{n+\frac{1}{2}}), \tag{2.5.2}$$

where the the convolution term approximation $R_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ needs to be evaluated carefully. For this, we consider the following piecewise linear function

$$\hat{\rho}^{n+\frac{1}{2}}(x) := \rho_{j-\frac{1}{2},+}^{n+\frac{1}{2}} + \frac{(x - x_{j-\frac{1}{2}})}{\Delta x}\left(\rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}} - \rho_{j-\frac{1}{2},+}^{n+\frac{1}{2}}\right) \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}).$$

Subsequently, the term $R_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ can be defined as

$$R_{j+\frac{1}{2}}^{n+\frac{1}{2}} := \sum_{k=0}^{N-1} \gamma_k \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}} + \frac{1}{\Delta x}\sum_{k=0}^{N-1} \chi_k \left(\rho_{j+k+\frac{3}{2},-}^{n+\frac{1}{2}} - \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}}\right),$$

where $\gamma_k \coloneqq \int_{k\Delta x}^{(k+1)\Delta x} w_\eta(y)\,\mathrm{d}y$ and $\chi_k \coloneqq \int_0^{\Delta x} y w_\eta(y+k\Delta x)\,\mathrm{d}y$ for $k = 0, \ldots, N-1$. This approximation of the convolution term is motivated by writing

$$\begin{aligned}
R(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}) &= \sum_{k=0}^{N-1} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} \rho(y, t^{n+\frac{1}{2}}) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&\approx \sum_{k=0}^{N-1} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} \hat{\rho}^{n+\frac{1}{2}}(y) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&= \sum_{k=0}^{N-1} \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&\quad + \frac{1}{\Delta x} \sum_{k=0}^{N-1} (\rho_{j+k+\frac{3}{2},-}^{n+\frac{1}{2}} - \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}}) \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} (y - x_{j+k+\frac{1}{2}}) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&= \sum_{k=0}^{N-1} \gamma_k \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}} + \frac{1}{\Delta x} \sum_{k=0}^{N-1} \chi_k \left(\rho_{j+k+\frac{3}{2},-}^{n+\frac{1}{2}} - \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}} \right).
\end{aligned}$$

Remark 2.5.1. Analogous to Remark 2.2.2, if the quadrature rule used to compute $R_{j+\frac{1}{2},\pm}^n$ in Step 1 is not exact for the given kernel function (i.e., if $\Delta x \sum_{k=0}^{N-1} w_\eta^k \neq 1$), then we replace w_η^k by $\tilde{w}_\eta^k = \dfrac{w_\eta^k}{Q_{\Delta x}}$, where we choose $Q_{\Delta x} \coloneqq \Delta x \sum_{k=0}^{N-1} w_\eta^k$.

2.6 Numerical results

In this section, we consider several test cases to demonstrate the performance of the proposed RK-2 and MH schemes described in Sections 2.2 and 2.5, respectively, by comparing it with the first-order Godunov-type scheme of [57, 89], which we denote by FO-Godunov. For all the test cases, we use the same CFL as that of RK-2 scheme, given in (2.2.13). Also, we choose $g(\rho) = \rho$ unless otherwise specified. Consider a uniform partition $\{I_j\}_{j=1}^M$ of the spatial domain $[a, b]$ with $\Delta x = \frac{b-a}{M}$. We will consider two types of boundary conditions: periodic and absorbing. In order to implement these boundary conditions, we will introduce ghost cells on either side of the domain. The ghost cell values, ρ_0^n and ρ_{M+j}^n for $j = 1, \ldots, N$, where $N = \eta/\Delta x$, are taken as follows. For periodic boundary conditions,

$$\rho_0^n = \rho_M^n \quad \text{and} \quad \rho_{M+j}^n = \rho_j^n \text{ for } j = 1, \ldots, N,$$

and for absorbing boundary conditions,

$$\rho_0^n = \rho_1^n \quad \text{and} \quad \rho_{M+j}^n = \rho_M^n \text{ for } j = 1, \ldots, N,$$

where $\{\rho_j^n\}_{j=1}^M$ denote the solution in real cells. In all the test cases, as the analytical solutions of (2.1.1) are not available, we use the RK-2 scheme (2.2.8) with fine mesh to generate reference solutions. These are used to determine the numerical errors and the experimental order of accuracy. The L^1-error for the cell average solution at time $t = t^n$ is given by

$$e(\Delta x) := \Delta x \sum_{j=1}^{M} |\rho_j^n - \rho_j^{n,ref}|,$$

where ρ_j^n and $\rho_j^{n,ref}$ are the cell averages of the numerical and the reference solutions, respectively. The experimental order of accuracy (E.O.A) is determined as

$$\Theta(\Delta x) := \log_2\Big(e(\Delta x)/e(\Delta x/2)\Big).$$

2.6.1 Mean downstream density model

In this part, we consider test cases with various initial data to solve the downstream density model equation given in (2.1.1):

$$\begin{aligned} \partial_t \rho + \partial_x\big(g(\rho)v(\rho * w_\eta)\big) &= 0, \quad x \in \mathbb{R},\, t \in (0,T], \\ \rho(0,x) &= \rho_0(x), \qquad x \in \mathbb{R}. \end{aligned}$$

Example 2.1. (smooth test case): To verify the order of accuracy of the proposed RK-2 and MH schemes, we consider the problem (2.1.1) with a smooth initial datum (see [57])

$$\rho_0(x) = 0.5 + 0.4 \sin \pi x. \tag{2.6.1}$$

The numerical solutions are computed in the domain $[-1, 1]$ with periodic boundary conditions. We choose $v(\rho) = 1 - \rho$ and the convolution parameter $\eta = 0.1$. Here, the reference solution is computed using a mesh size of $\Delta x = \frac{1}{1280}$. The solutions are computed up to time $T = 0.15$ for three different kernel functions $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ with time steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$, $\Delta t = \frac{\Delta x}{2+20\Delta x}$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$ respectively. From Table 2.1, we observe that both the RK-2 and MH schemes exhibit the desired experimental order of accuracy. In Figure 2.1, we provide the L^1 error versus CPU time plots for the RK-2 and MH schemes, corresponding to the initial data (2.6.1) and considering the three kernel functions mentioned above. Here, we use the mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. The results indicate that the MH scheme is computationally more efficient when compared to the RK-2 scheme.

Example 2.2. We consider the problem (2.1.1) with a discontinuous initial datum as given in [97],

$$\rho_0(x) = \begin{cases} 0.8, & \text{if} - 0.5 < x < -0.1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.6.2}$$

$w_\eta(x)$		FO-Godunov		RK-2		MH	
	Δx	L^1-error	E.O.A.	L^1-error	E.O.A.	L^1-error	E.O.A.
$\frac{1}{\eta}$	0.05	0.013011	-	0.001686	-	0.001474	-
	0.025	0.006478	1.006117	0.000463	1.862867	0.000374	1.978590
	0.0125	0.003199	1.017847	0.000122	1.924356	9.635912e-05	1.956891
	0.00625	0.001591	1.007311	3.240261e-05	1.914917	2.486745e-05	1.954162
	0.003125	0.000794	1.001897	8.062984e-06	2.006724	5.996242e-06	2.052127
$\frac{2(\eta-x)}{\eta^2}$	0.05	0.014857	-	0.004348	-	0.001489	-
	0.025	0.007085	1.068269	0.001151	1.917093	0.000369	2.009029
	0.0125	0.003436	1.044055	0.000299	1.943794	9.558809e-05	1.952366
	0.00625	0.001687	1.026122	7.636725e-05	1.970390	2.448845e-05	1.964729
	0.003125	0.000835	1.013861	1.880892e-05	2.021536	5.907713e-06	2.051429
$\frac{3(\eta^2-x^2)}{2\eta^3}$	0.05	0.014294	-	0.003977	-	0.001452	-
	0.025	0.006894	1.051945	0.001024	1.956857	0.000366	1.987019
	0.0125	0.003358	1.037732	0.000265	1.949414	9.510829e-05	1.945937
	0.00625	0.001654	1.021031	6.804842e-05	1.962939	2.450048e-05	1.956760
	0.003125	0.000820	1.011442	1.679244e-05	2.018749	5.900419e-06	2.053920

Table 2.1: Example 2.1. L^1-errors and E.O.A. obtained using the FO-Godunov, RK-2 and MH schemes to solve the problem (2.1.1) with smooth initial condition (2.6.1) and three different kernel functions: $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$ where $\eta = 0.1$. Numerical solutions are computed up to time $T = 0.15$.

described in the computational domain $[-1, 1]$ and using absorbing boundary conditions. The velocity and convolution parameters are set as $v(\rho) = 1 - \rho$ and $\eta = 0.1$, respectively. The numerical solutions are computed at time T $= 0.5$ with a mesh size of $\Delta x = 0.01$ for two different kernel functions: $w_\eta(x) = \frac{1}{\eta}$ and $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$, with the respective time steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$ and $\Delta t = \frac{\Delta x}{2+20\Delta x}$. The reference solutions are computed using a mesh-size of $\Delta x = \frac{1}{2560}$. The results are depicted in Figure 2.2(a) and Figure 2.2(c), respectively. Also, zoomed images of the region $[0.3, 0.55] \times [-0.01, 0.19]$ for both the cases are given in Figure 2.2(b) and Figure 2.2(d), respectively. It is observed that both the RK-2 and MH schemes provide better resolution than the first-order Godunov type scheme. Moreover, the RK-2 and MH solutions are comparable, with the MH solution giving slightly better resolution near the right discontinuity as seen in Figures 2.2(b) and 2.2(d). The L^1 error versus CPU time plots for the RK-2 and MH schemes corresponding to the initial datum (2.6.2), for the kernel functions $w_\eta(x) = \frac{1}{\eta}$ and $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ are given in Figure 2.5(a) and Figure 2.5(b), respectively. The solutions are computed with mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. The results show that the MH scheme is computationally more efficient when compared to the RK-2 scheme.

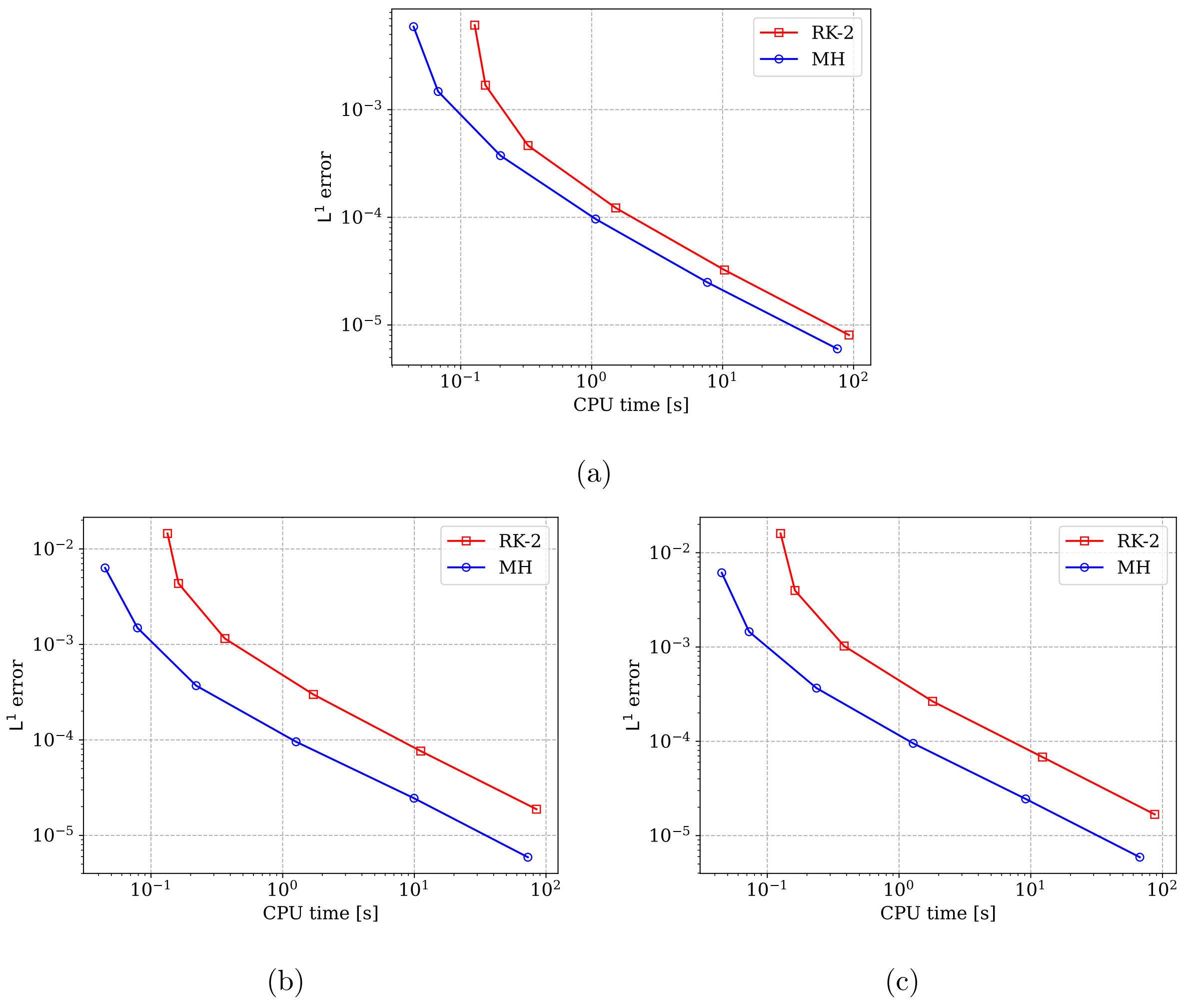


Figure 2.1: Example 2.1. L^1 error versus CPU time plots in log–log scale for the RK-2 and MH schemes to solve the problem (2.1.1) with initial datum (2.6.1) and three different kernel functions (a) $w_\eta(x) = \frac{1}{\eta}$, (b) $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ and (c) $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$. The solutions are computed at time T=0.15.

Example 2.3. In this example we consider the same initial datum (2.6.2), particularly to see the behaviour of the solutions at two different times. Here, we use this initial condition to simulate the scalar problem (2.1.1) in the computational domain $[-1, 2]$. The velocity v is given by $v(\rho) = v_{\max}(1-\rho)$ with $v_{\max} = 0.8$, and the convolution kernel $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ with $\eta = 0.3$. Numerical solutions are computed at two different time levels, $T = 1.0, 2.0$ with a mesh size of $\Delta x = 0.0125$ and time step $\Delta t = \frac{\Delta x}{2+\frac{20}{3}\Delta x}$ using absorbing boundary conditions. The results are shown in Figure 2.3. For the reference solution, we use the RK-2 scheme with a fine mesh of size $\Delta x = 0.0025$. It is observed that at both times the RK-2 and MH schemes give better resolution than the first-order scheme.

Example 2.4. In this example, we evolve (2.1.1) for a quadratic kernel function with the initial datum (2.6.2) in the computational domain $x \in [-1, 1]$. Further, we choose:

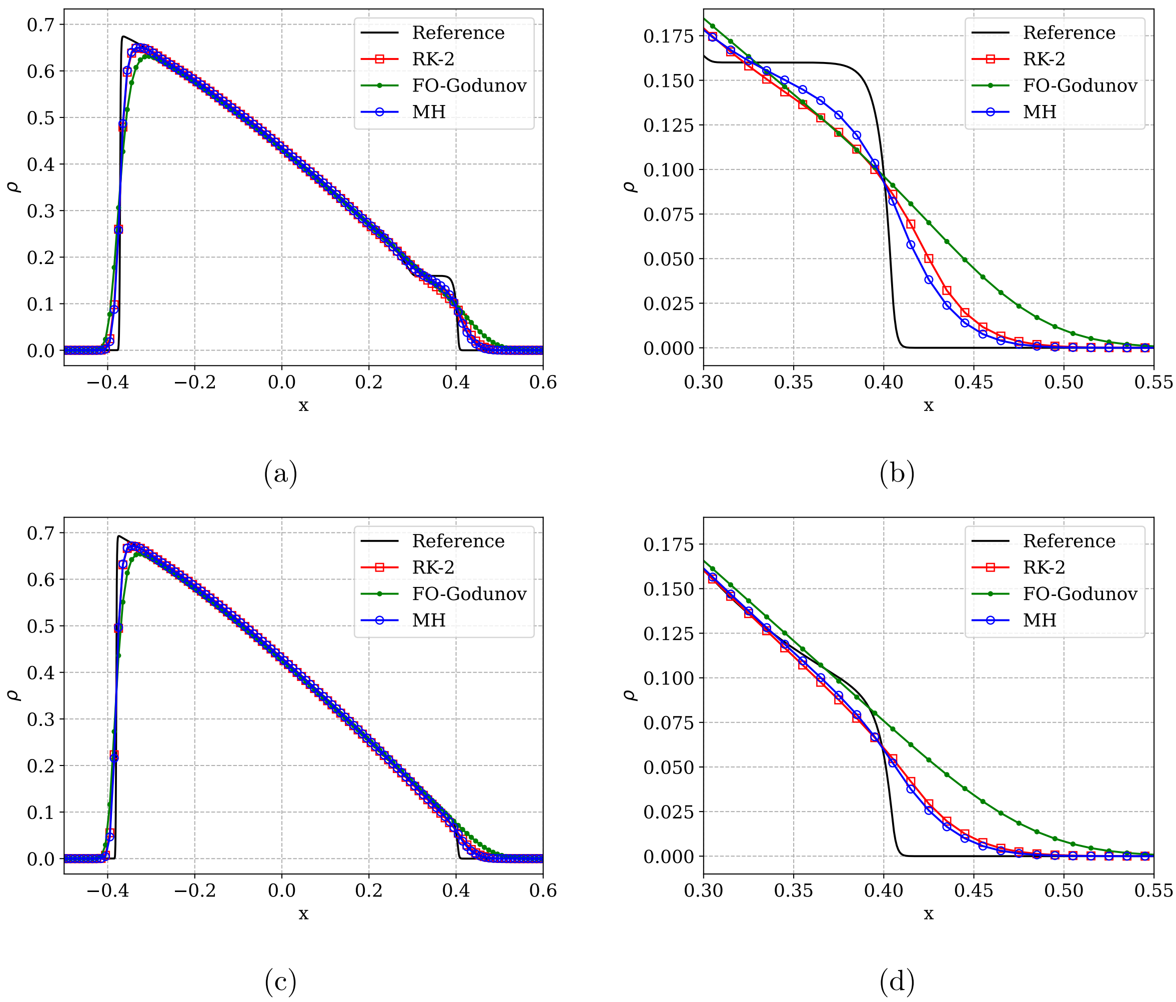
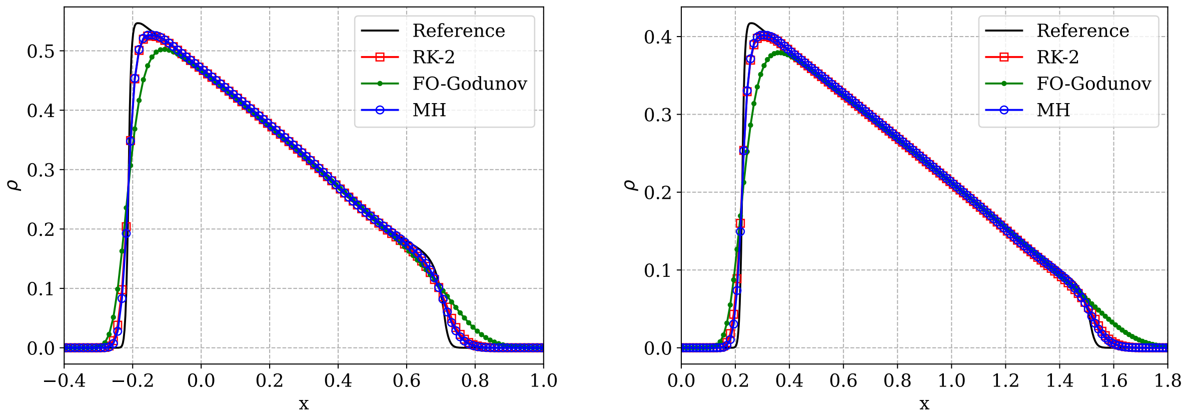


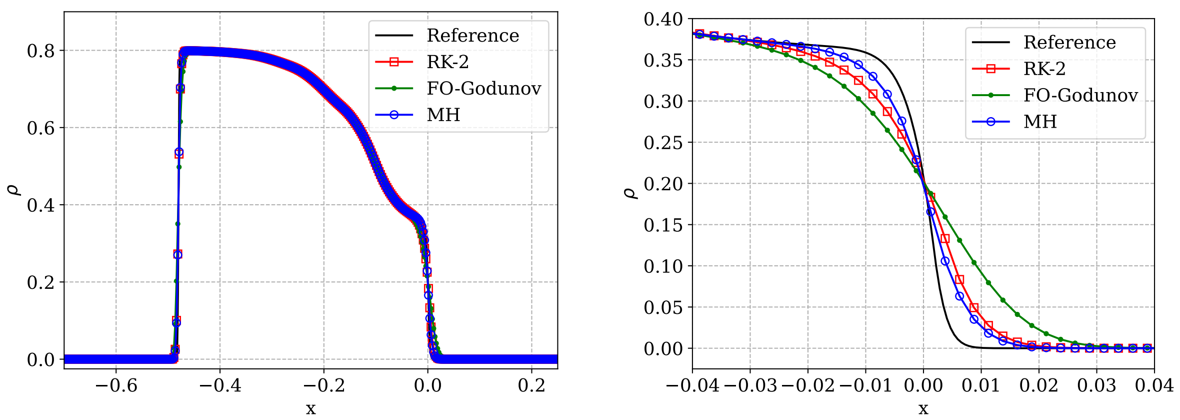
Figure 2.2: Example 2.2. Numerical solutions of (2.1.1) at time T = 0.5 with the initial condition (2.6.2) for two different kernel functions (a) $w_\eta(x) = \frac{1}{\eta}$ and (c) $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$, where $\eta = 0.1$. (b) Zoomed image of the region $[0.3, 0.55] \times [-0.01, 0.19]$ in (a), (d) Zoomed image of the region $[0.3, 0.55] \times [-0.01, 0.19]$ in (c). The velocity function $v(\rho) = 1 - \rho$. A mesh size of $\Delta x = 0.01$ is chosen with the times steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$ and $\Delta t = \frac{\Delta x}{2+20\Delta x}$ for the two kernels (a) and (c), respectively.

$v(\rho) = 1-\rho$, $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = 0.0025$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. Numerical solutions are computed at time $T = 0.1$ using absorbing boundary conditions and are given in Figure 2.4. Reference solution is obtained using the RK-2 scheme with a mesh size of $\Delta x = 0.000625$. Figure 2.4(b) is the enlarged view of the region $[-0.04, 0.04] \times [-0.025, 0.400]$ in Figure 2.4(a). It is observed that the RK-2 and MH solutions give a better resolution compared to the Godunov-type scheme. The L^1 error versus CPU time plot for the RK-2 and MH schemes corresponding to the initial datum (2.6.2) with the kernel function $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$ is given in Figure 2.5(c). The solutions are computed using the mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. Here also, we observe that the MH scheme is more efficient compared to the RK-2 scheme.



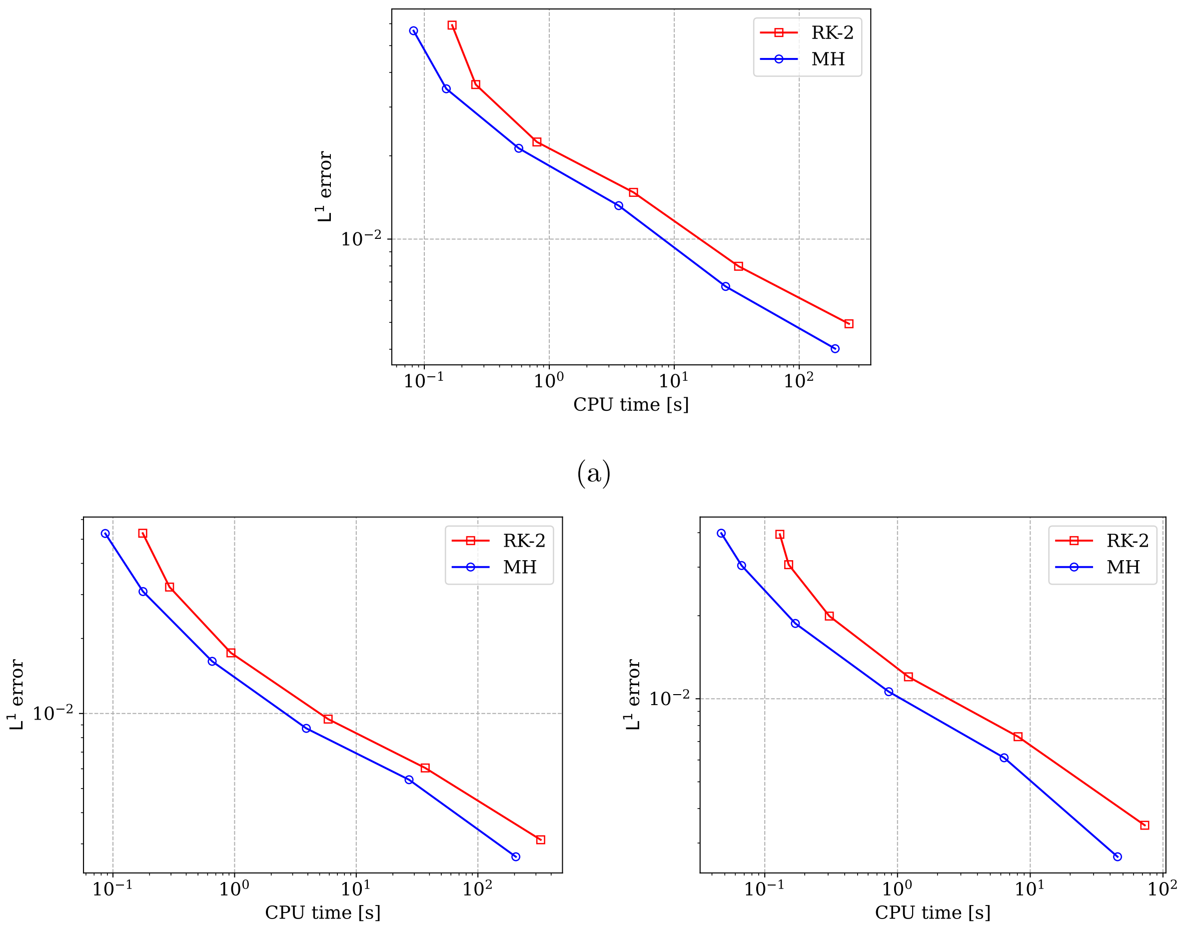
(a) (b)

Figure 2.3: Example 2.3. Numerical solutions of (2.1.1) with the initial condition (2.6.2) at two different times: (a) $T = 1.0$ and (b) $T = 2.0$. Mesh size $\Delta x = 0.0125$, $\Delta t = \frac{\Delta x}{2+\frac{20}{3}\Delta x}$, $v(\rho) = 0.8(1-\rho)$, $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$, $\eta = 0.3$.



(a) (b)

Figure 2.4: Example 2.4. (a) Numerical solutions of (2.1.1) with initial datum (2.6.2) at time $T = 0.1$, where $v(\rho) = 1-\rho$, $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = \frac{1}{400}$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. (b) Enlarged view of the region $[-0.04, 0.04] \times [-0.025, 0.400]$ in plot (a).



(a)

(b)

(c)

Figure 2.5: Examples 2.2 and 2.4. L^1 error versus CPU time plots in log–log scale for the RK-2 and MH schemes to solve the problem (2.1.1) with the discontinuous initial data (2.6.2) as described in: (a) Example 2.2 with $w_\eta(x) = \frac{1}{\eta}$ at time $T = 0.5$, (b) Example 2.2 with $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ at time $T = 0.5$ and (c) Example 2.4 with $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ at time $T = 0.1$.

2.6.2 Mean downstream velocity model

We consider the initial value problem for the downstream velocity model (A.2.1) outlined in Appendix A.2

$$\partial_t \rho + \partial_x \Big(g(\rho) \big(v(\rho) * w_\eta \big) \Big) = 0, \quad x \in \mathbb{R}, t \in (0, T],$$
$$\rho(x, 0) = \rho_0(x), \qquad x \in \mathbb{R},$$

with three test cases. The first test case involves a smooth function and will be used to confirm the order of accuracy of the schemes. The remaining test cases involve initial data with discontinuities and will be used to compare the performance of first-order and second-order schemes.

Example 2.5. (smooth test case): To verify the desired order of accuracy of the proposed RK-2 and MH schemes, we consider a test case with the same smooth initial datum as in (2.6.1) to evolve (A.2.1), in the computational domain $x \in [-1, 1]$ together with periodic boundary conditions. The velocity function v is taken as $v(\rho) = 1 - \rho^2$ and $g(\rho) = \rho^2$. Numerical solutions are computed upto time $T = 0.15$ for the kernel functions $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$ where $\eta = 0.1$ and the respective time steps are $\Delta t = \frac{\Delta x}{2+10\Delta x}$, $\Delta t = \frac{\Delta x}{2+20\Delta x}$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. Here, we use a reference solution generated with a mesh of size $\Delta x = \frac{1}{1280}$. The results are tabulated in Table 2.2.

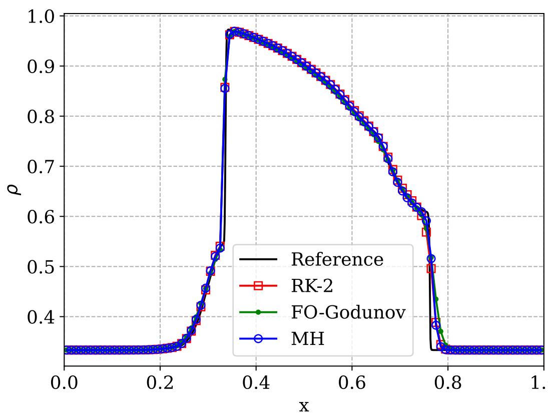
$w_\eta(x)$		FO-Godunov		RK-2		MH	
	Δx	L^1-error	E.O.A.	L^1-error	E.O.A.	L^1-error	E.O.A.
	0.05	0.014125	-	0.001002	-	0.001230	-
	0.025	0.007099	0.992521	0.000280	1.839409	0.000330	1.895510
	0.0125	0.003542	1.002772	7.413608e-05	1.917842	8.579633e-05	1.947201
$\frac{1}{\eta}$	0.00625	0.001767	1.003282	1.893899e-05	1.968816	2.169048e-05	1.983853
	0.003125	0.000882	1.002633	4.501158e-06	2.072991	5.206407e-06	2.058702
	0.05	0.016040	-	0.002595	-	0.001413	-
	0.025	0.007745	1.050226	0.000744	1.801440	0.000355	1.991035
	0.0125	0.003780	1.034750	0.000197	1.912197	8.939415e-05	1.992234
$\frac{2(\eta-x)}{\eta^2}$	0.00625	0.001864	1.020026	5.068842e-05	1.964803	2.266346e-05	1.979811
	0.003125	0.000925	1.010639	1.250372e-05	2.019297	5.805551e-06	1.964863
	0.05	0.015555	-	0.002308	-	0.001331	-
	0.025	0.007569	1.039206	0.000621	1.893337	0.000340	1.965531
	0.0125	0.003716	1.026320	0.000161	1.939921	8.587823e-05	1.988833
$\frac{3(\eta^2-x^2)}{2\eta^3}$	0.00625	0.001837	1.015781	4.141757e-05	1.967297	2.176546e-05	1.980251
	0.003125	0.000913	1.008295	1.024745e-05	2.014977	5.485051e-06	1.988463

Table 2.2: Example 2.5. L^1-errors and E.O.A. obtained for the RK-2 and MH schemes to solve the problem (A.2.1) with smooth initial condition (2.6.1) at $T = 0.15$, $\eta = 0.1$ and for the kernel functions: $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta-x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$.

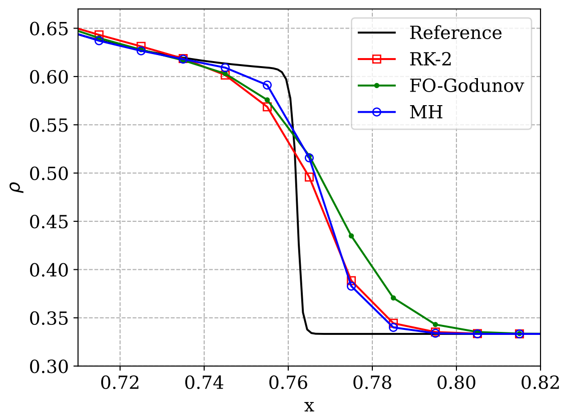
Example 2.6. (Non-linear velocity): In this test case, we choose a discontinuous initial datum (see [89]), for the problem (A.2.1)

$$\rho_0(x) = \begin{cases} 1, & \text{if } 1/3 < x < 2/3, \\ \frac{1}{3}, & \text{otherwise,} \end{cases} \tag{2.6.3}$$

in the domain $[0, 1]$. Further, we choose the convolution kernel: $w_\eta(x) = \frac{3(\eta^2-x^2)}{2\eta^3}$ with $\eta = 0.1$, velocity function $v(\rho) = 1 - \rho^2$, $g(\rho) = \rho^2$ and time step $\Delta t = \frac{\Delta x}{2+15\Delta x}$ with a mesh



(a)



(b)

Figure 2.6: Example 2.6. (a) Simulation of (A.2.1) at time $T = 0.1$ with $v(\rho) = 1 - \rho^2$, $g(\rho) = \rho^2$, $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = 0.01$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. (b) Enlarged view of the region $[0.71, 0.82] \times [0.30, 0.67]$ in plot (a).

size of $\Delta x = 0.01$. By imposing periodic boundary conditions, numerical solutions are computed at time $T = 01.$ and time-step $\Delta t = \frac{\Delta x}{2+15\Delta x}$. The results are plotted in Figure 2.6, where the reference solution is computed with a mesh size of $\Delta x = \frac{1}{1000}$. As illustrated in Figure 2.6(b), an enlarged view of the region $[0.71, 0.82] \times [0.3, 0.67]$ reveals that the RK-2 and MH schemes provide higher resolution than the first-order Godunov-type scheme.

Example 2.7.(Non-linear velocity): We now consider an example with a non-linear velocity function $v(\rho) = 1 - \rho^5$, described in [89]. The problem (A.2.1) is simulated in the computational domain $[0, 1]$ with the initial datum (2.6.3) and the kernel function $w_\eta(x) = \frac{1}{\eta}$, where $\eta = 0.1$. Numerical solutions are computed at time $T = 0.05$ with periodic boundary conditions and a mesh of size $\Delta x = \frac{1}{100}$. The time-step is chosen as $\Delta t = \frac{\Delta x}{2+10\Delta x}$. The results can be seen in Figure 2.7, where the reference solution is computed with a fine mesh of size $\Delta x = \frac{1}{1000}$. The plots indicate that the second-order RK-2 and MH schemes produce better resolution than the Godunov-type scheme. For a better visualization, we have provided an enlarged view of the region $[0.66, 0.77] \times [0.3, 0.8]$ in Figure 2.7(b).

2.7 Concluding remarks

We have conducted a study on the numerical approximation of a class of non-local conservation laws modelling traffic-flow problems, with more emphasis on the convergence analysis. We use a MUSCL-type spatial reconstruction and strong stability preserving Runge-Kutta time stepping to derive a second-order scheme, denoted by RK-2. The resulting scheme is shown to converge to a weak solution of the given problem. In addition,

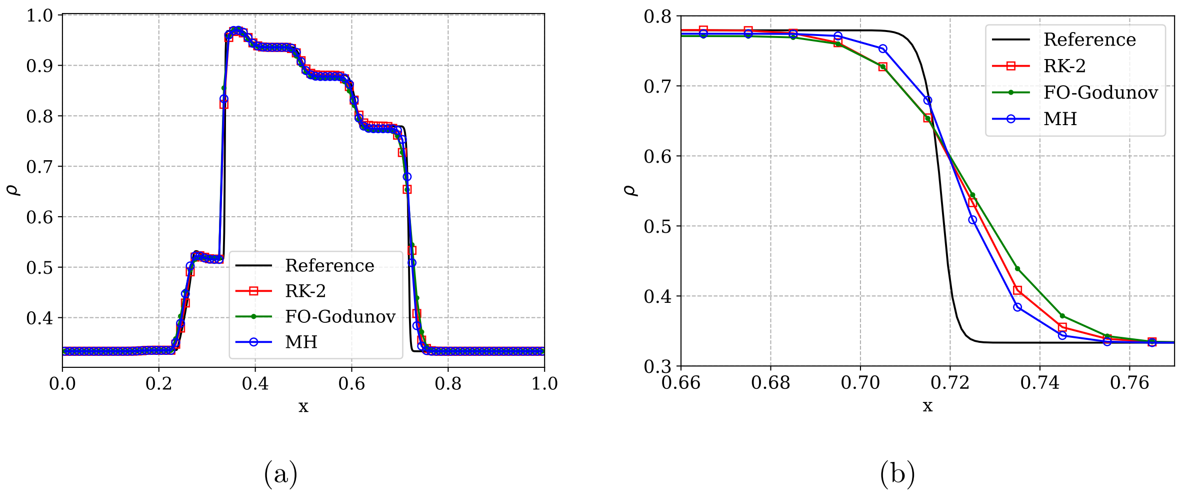


Figure 2.7: Example 2.7. Behaviour of first-order and second-order schemes with non-linear velocity $v(\rho) = 1 - \rho^5$. (a) Numerical solution of (A.2.1) at time $T = 0.1$ with $w_\eta(x) = \frac{1}{\eta}$, $\eta = 0.1$, $\Delta x = 0.01$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. (b) Enlarged view of the region $[0.66, 0.77] \times [0.3, 0.8]$ in (a).

using a space-step dependent slope limiter we show that the scheme converges to the unique entropy solution. Further, we have proposed a MUSCL-Hancock type (MH) second-order scheme which requires only one intermediate stage in the time evolution, unlike the RK schemes. We observe that both the second-order schemes produce stable and accurate solutions. Additionally, we notice that the MH scheme gives slightly better results compared to the RK-2 scheme, for example see Figures 2.2(b), 2.2(d), 2.6(b) and 2.7(b) of Examples 2, 6 and 7. Further, the L^1 error versus CPU time plots in Figures 2.1 and 2.5 indicate that the MH-2 scheme is computationally more efficient in comparison to the RK-2 scheme.

3

MUSCL-Hancock Scheme for Non-Local Conservation Laws

In Chapter 2, we developed second-order schemes for non-local conservation laws with monotone kernels and downstream convolution, specifically tailored for traffic flow modeling. This chapter addresses a more general class of non-local conservation laws (referred to as Model 2 in Chapter 1, Section 1.1), for which we present a provably convergent, single-stage, second-order MUSCL–Hancock (MH)-type scheme. The MH method, originally proposed in [163], combines MUSCL-type spatial reconstruction [162] with a predictor–corrector time integration based on the midpoint rule. Starting from the reconstructed values at time t^n, the predictor step employs a Taylor expansion to compute interface values at $t^{n+\frac{1}{2}}$, which are then used in the corrector step to evaluate the numerical fluxes. Compared to conventional two-stage Runge–Kutta-based MUSCL schemes, the MH approach is computationally more efficient, requiring only one spatial reconstruction and one numerical flux evaluation per time step. Owing to its simplicity and efficiency, the MH method has been widely adopted; see [31, 168, 106, 49, 158, 159] for recent advances and applications.

The principal challenge in designing a MUSCL–Hancock-type scheme for non-local conservation laws lies in the accurate discretization of the non-local term in the flux function. Achieving second-order accuracy while enabling a rigorous convergence analysis necessitates careful treatment of this term in both the predictor and corrector steps. In Chapter 2, Section 2.5, we introduced a MUSCL–Hancock-type scheme for a class of non-local traffic flow models with piecewise smooth, non-increasing convolution kernels. While that method performs well numerically (as demonstrated in Chapter 2, Section 2.6), a rigorous convergence proof remains elusive. In this chapter, we formulate a novel discretization of the convolution term and propose a new MH-type scheme for problems of the type Model 2 (described in Chapter 1, Section 1.1), originally studied in [19, 14]. Specifically, the predictor step computes the interface convolution values using a piecewise linear reconstruction with cellwise slopes chosen appropriately. In the corrector step, the convolutions at the intermediate time level $t^{n+\frac{1}{2}}$ are approximated using a suitable quadrature rule.

As a major contribution of this work, we establish the convergence of the proposed scheme to the unique entropy solution of the problem under consideration. To this end, we first reformulate the scheme in a suitable form and prove the positivity-preserving property and L^∞ stability under an appropriate CFL condition. We then derive a total variation bound and a time-continuity estimate. These analyses present several difficulties, which we overcome using the properties of the slope-limiter and the novel discrete convolutions in the predictor step. Leveraging the derived estimates, Kolmogorov's compactness theorem provides us the existence of a convergent subsequence of approximate solutions. To ensure that the entire scheme converges to the entropy solution of the underlying problem, we introduce a mesh-dependent modification to the slope limiter, following the strategy in [166, 167]. The resulting scheme is then shown to converge to the unique entropy solution. We also provide numerical examples to support the theoretical results and to compare the proposed method with a first-order scheme and a conventional second-order MUSCL-type scheme based on Runge-Kutta time integration.

The remainder of the chapter is structured as follows. In Section 3.1, we present the necessary preliminaries for the class of non-local conservation laws considered in this chapter. Section 3.2 describes the proposed numerical scheme in detail. Uniform a priori estimates for the scheme are established in Section 3.3. In Section 3.4, we prove the convergence of the scheme to the unique entropy solution. Numerical experiments are presented in Section 3.5. Some essential but technically lengthy estimates are deferred to Appendix B.1, while the construction of a standard MUSCL–Runge–Kutta scheme is detailed in Appendix B.2.

3.1 Non-local conservation laws

We are interested in the initial value problem for one dimensional non-local conservation laws considered in [14, 19]:

$$\begin{aligned} \partial_t \rho + \partial_x f(\rho, A(t,x)) = 0, \qquad & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), \qquad & x \in \mathbb{R}, \end{aligned} \tag{3.1.1}$$

where $\mu : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\rho : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ denote the convolution kernel, the given flux function and the unknown quantity, respectively. Here, the convolution term $\mu * \rho$ is defined as

$$A(t,x) := \mu * \rho(t,x) = \int_{-\infty}^{\infty} \mu(x-y)\rho(t,y)\,\mathrm{d}y. \tag{3.1.2}$$

3.1.1 Hypotheses

The functions f and μ in (3.1.1) and (3.1.2) are assumed to satisfy the following hypothesis:

H1. *For all $A \in \mathbb{R}$,*

(i) $f \in \mathrm{C}^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, $\quad \partial_\rho f \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}; \mathbb{R})$,
(ii) $f(0, A) = 0$.

H2. *There exists an $M > 0$ such that for all ρ, A in the respective domains,*

$$|\partial_A f|,\ |\partial^2_{AA} f| \leq M|\rho|.$$

H3. $\partial_\rho f \in \mathrm{W}^{1,\infty}(\mathbb{R} \times \mathbb{R}; \mathbb{R})$.

H4. $\mu \in (\mathrm{C}^2_c \cap \mathrm{W}^{2,\infty})(\mathbb{R}; \mathbb{R})$.

Solutions to non-linear conservation laws need not be smooth in general, even in the case when the initial datum is smooth. Therefore, we consider the weak/entropy formulations of the solutions to (3.1.1) defined below.

3.1.2 Weak and entropy solutions

Definition 3.1.1. (Weak solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R}; \mathbb{R})$, $T > 0$, is a weak solution of (3.1.1) if

$$\int_0^T \int_{-\infty}^{+\infty} \big(\rho \partial_t \varphi + f(\rho, A(t,x))\partial_x \varphi\big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} \rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0 \tag{3.1.3}$$

for all $\varphi \in \mathrm{C}^1_c([0,T) \times \mathbb{R}; \mathbb{R})$.

Next, following the definition in [33], we define an entropy solution to the problem (3.1.1) as follows:

Definition 3.1.2. (Entropy solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R}; \mathbb{R})$, $T > 0$, is an entropy weak solution of (3.1.1) if

$$\int_0^T \int_{-\infty}^{+\infty} \Big(|\rho - \kappa| \partial_t \varphi + \operatorname{sgn}(\rho - \kappa)\big(f(\rho, A(t,x)) - f(\kappa, A(t,x))\big)\partial_x \varphi \\ - \operatorname{sgn}(\rho - \kappa)\partial_A f(\kappa, A(t,x))\partial_x(A(t,x))\varphi\Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa|\varphi(0,x)\,\mathrm{d}x \geq 0 \tag{3.1.4}$$

for all $\varphi \in \mathrm{C}_c^1([0,T) \times \mathbb{R}; \mathbb{R}^+)$ and $\kappa \in \mathbb{R}$, where sgn is the sign function.

We note that the weak solution to (3.1.1) that satisfies the entropy inequality (3.1.4) is unique, as observed in [33, 34].

3.2 A MUSCL-Hancock-type second-order scheme

We discretize the spatial domain into Cartesian grids with a uniform mesh of size Δx. The spatial domain is now a union of cells of the form $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ where $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x$ and $x_j = j\Delta x$. We fix $T > 0$ and the time-step is denoted by Δt and $t^n = n\Delta t$ for $n \in \mathbb{N}$, $\lambda = \frac{\Delta t}{\Delta x}$. The initial datum ρ_0 is discretized as

$$\rho_j^0 = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho_0(x)\,\mathrm{d}x \quad \text{for } j \in \mathbb{Z}.$$

Given the cell-average solutions $\{\rho_j^n\}_{j\in\mathbb{Z}}$ at the $n-$th time-level, the first step is to obtain a piecewise linear reconstruction as follows

$$\tilde{\rho}_\Delta^n(x) := \rho_j^n + \frac{(x - x_j)}{\Delta x}\sigma_j^n \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \tag{3.2.1}$$

where the slopes σ_j^n are computed as

$$\sigma_j^n = 2\theta \operatorname{minmod}\left((\rho_j^n - \rho_{j-1}^n),\ \frac{1}{2}(\rho_{j+1}^n - \rho_{j-1}^n),\ (\rho_{j+1}^n - \rho_j^n) \right), \tag{3.2.2}$$

with $\theta \in [0, 0.5]$. The left and right face values (at the interface $x = x_{j+\frac{1}{2}}$) of the reconstructed polynomial are given by

$$\rho_{j+\frac{1}{2}}^{n,-} = \rho_j^n + \frac{\sigma_j^n}{2}, \quad \rho_{j+\frac{1}{2}}^{n,+} = \rho_{j+1}^n - \frac{\sigma_{j+1}^n}{2}. \tag{3.2.3}$$

Next, a finite volume integration of the conservation law (3.1.1) in the domain $[t^n, t^{n+1}] \times [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ yields

$$
\begin{aligned}
&\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(t^{n+1}, x)\,\mathrm{d}x \\
&\quad = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(t^{n}, x)\,\mathrm{d}x \\
&\qquad - \left(\int_{t^n}^{t^{n+1}} f(\rho(t, x_{j+\frac{1}{2}}), A(t, x_{j+\frac{1}{2}}))\,\mathrm{d}t - \int_{t^n}^{t^{n+1}} f(\rho(t, x_{j-\frac{1}{2}}), A(t, x_{j-\frac{1}{2}}))\,\mathrm{d}t\right) \\
&\quad \approx \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho(t^{n}, x)\,\mathrm{d}x \\
&\qquad - \Delta t \left(f(\rho(t^{n+\frac{1}{2}}, x_{j+\frac{1}{2}}), A(t^{n+\frac{1}{2}}, x_{j+\frac{1}{2}})) - f(\rho(t^{n+\frac{1}{2}}, x_{j-\frac{1}{2}}), A(t^{n+\frac{1}{2}}, x_{j-\frac{1}{2}}))\right),
\end{aligned}
\tag{3.2.4}
$$

where we have used midpoint quadrature rule in approximating the flux integral. From (3.2.4), we formulate a MUSCL-Hancock type finite volume scheme as follows

$$
\rho_j^{n+1} = \rho_j^n - \lambda \left[F_{j+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right], \tag{3.2.5}
$$

where for an appropriately chosen numerical flux F, we define

$$
F_{j+\frac{1}{2}}^{n+\frac{1}{2}} := F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}}), \; j \in \mathbb{Z},
$$

and the mid-time density values are obtained using Taylor series expansions as follows

$$
\begin{aligned}
\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} &= \rho_{j+\frac{1}{2}}^{n,-} - \frac{\lambda}{2}\left(f(\rho_{j+\frac{1}{2}}^{n,-}, A_{j+\frac{1}{2}}^{n,-}) - f(\rho_{j-\frac{1}{2}}^{n,+}, A_{j-\frac{1}{2}}^{n,+})\right), \\
\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+} &= \rho_{j-\frac{1}{2}}^{n,+} - \frac{\lambda}{2}\left(f(\rho_{j+\frac{1}{2}}^{n,-}, A_{j+\frac{1}{2}}^{n,-}) - f(\rho_{j-\frac{1}{2}}^{n,+}, A_{j-\frac{1}{2}}^{n,+})\right), \quad j \in \mathbb{Z}.
\end{aligned}
\tag{3.2.6}
$$

The terms $A_{j\pm\frac{1}{2}}^{n+\frac{1}{2}}$ in (3.2.5) and $A_{j\mp\frac{1}{2}}^{n,\pm}$ in (3.2.6) are suitable approximations of the convolution terms $A(t^{n+\frac{1}{2}}, x_{j\pm\frac{1}{2}})$ and $A(t^n, x_{j\mp\frac{1}{2}}^{n,\pm})$, respectively. These approximations are elaborated in the following section.

3.2.1 Approximation of convolution terms

To begin with, using Taylor series expansions we write

$$
\begin{aligned}
A(t^n, x_{j-\frac{1}{2}}) &= A(t^n, x_j) - \frac{\Delta x}{2}\partial_x A(t^n, x_j) + \mathcal{O}(\Delta x^2), \\
A(t^n, x_{j+\frac{1}{2}}) &= A(t^n, x_j) + \frac{\Delta x}{2}\partial_x A(t^n, x_j) + \mathcal{O}(\Delta x^2).
\end{aligned}
\tag{3.2.7}
$$

Next, we approximate the quantities on the right hand side of (3.2.7) using suitable quadrature rules. To this end, using midpoint quadrature rule, we first write

$$
\begin{aligned}
A(t^n, x_j) &= \int_{\mathbb{R}} \mu(x_j - y)\rho(t^n, y)\,\mathrm{d}y \\
&= \sum_{l\in\mathbb{Z}} \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \mu(x_j - y)\rho(t^n, y)\,\mathrm{d}y \approx \Delta x \sum_{l\in\mathbb{Z}} \mu_{j-l}\, \rho_l^n =: A_j^n,
\end{aligned}
\tag{3.2.8}
$$

where $\mu_j := \mu(j\Delta x)$, $j \in \mathbb{Z}$. Further, using the central difference approximation to the derivative, we write

$$\begin{aligned}\Delta x \partial_x A(t^n, x_j) &= \frac{1}{2}\left(A(t^n, x_{j+1}) - A(t^n, x_{j-1})\right) + \mathcal{O}(\Delta x^3) \\ &\approx \frac{1}{2}(A^n_{j+1} - A^n_{j-1}) + \mathcal{O}(\Delta x^3).\end{aligned} \tag{3.2.9}$$

Plugging in the approximations (3.2.8) and (3.2.9) to (3.2.7), we obtain

$$\begin{aligned}A(t^n, x_{j-\frac{1}{2}}) &\approx A^n_j - \frac{1}{4}(A^n_{j+1} - A^n_{j-1}) =: \hat{A}^{n,+}_{j-\frac{1}{2}}, \\ A(t^n, x_{j+\frac{1}{2}}) &\approx A^n_j + \frac{1}{4}(A^n_{j+1} - A^n_{j-1}) =: \hat{A}^{n,-}_{j+\frac{1}{2}}.\end{aligned} \tag{3.2.10}$$

Motivated by (3.2.10) and using the parameter θ from (3.2.2), we redefine the left and right approximate convolutions in the cell $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ as follows

$$A^{n,-}_{j+\frac{1}{2}} := A^n_j + \frac{s^n_j}{2}, \quad A^{n,+}_{j-\frac{1}{2}} := A^n_j - \frac{s^n_j}{2}, \tag{3.2.11}$$

where $s^n_j := 2\theta \dfrac{(A^n_{j+1} - A^n_{j-1})}{2}$ for $\theta \in [0, 0.5]$.

Remark 3.2.1. If we choose $\theta = 0.5$, the approximations (3.2.11) and (3.2.10) coincide – i.e., $A^{n,-}_{j+\frac{1}{2}} = \hat{A}^{n,-}_{j+\frac{1}{2}}$, and $A^{n,+}_{j-\frac{1}{2}} = \hat{A}^{n,+}_{j-\frac{1}{2}}$. On the other hand, if we choose $\theta = 0$, then $s^n_j = 0$ for all $j \in \mathbb{Z}$ and consequently $A^{n,+}_{j-\frac{1}{2}} = A^{n,-}_{j+\frac{1}{2}} = A^n_j$.

Once the mid-time density values $\rho^{n+\frac{1}{2},\mp}_{j\pm\frac{1}{2}}$ are computed using (3.2.6), the mid-time convolution approximation $A^{(n+\frac{1}{2})}_{j+\frac{1}{2}} \approx A(t^{n+\frac{1}{2}}, x_{j+\frac{1}{2}})$ in (3.2.5) is approximated using the trapezoidal quadrature rule as follows

$$\begin{aligned}A(t^{n+\frac{1}{2}}, x_{j+\frac{1}{2}}) &= \sum_{l\in\mathbb{Z}} \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \mu(x_{j+\frac{1}{2}} - y)\rho(t^{n+\frac{1}{2}}, y)\,\mathrm{d}y \\ &\approx \frac{\Delta x}{2} \sum_{l\in\mathbb{Z}} \left[\mu_{j+1-l}\, \rho^{n+\frac{1}{2},+}_{l-\frac{1}{2}} + \mu_{j-l}\, \rho^{n+\frac{1}{2},-}_{l+\frac{1}{2}}\right] =: A^{n+\frac{1}{2}}_{j+\frac{1}{2}}.\end{aligned} \tag{3.2.12}$$

3.2.2 Numerical flux

Throughout this chapter, we consider the scheme (3.2.5) with a Lax-Friedrichs type numerical flux, which we define following the formulation in [14, 19, 15] as described below

$$F(u, v, A) = \frac{f(u, A) + f(v, A)}{2} - \frac{\alpha(v-u)}{2\lambda}, \quad \text{for } u, v \in \mathbb{R}, \tag{3.2.13}$$

and for a fixed coefficient $\alpha \in (0, \frac{8}{27})$.

Finally, for a fixed mesh size Δx, we denote by $\rho_\Delta(t, x)$ the piecewise constant approximate solution obtained from the scheme (3.2.5):

$$\rho_\Delta(t, x) := \rho^n_j \quad \text{for} \quad (t, x) \in [t^n, t^{n+1}) \times [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \quad \text{for } n \in \mathbb{N} \text{ and } j \in \mathbb{Z}. \tag{3.2.14}$$

Remark 3.2.2. Setting $\theta = 0$ in (3.2.2) and (3.2.11) leads to $\sigma_j^n = s_j^n = 0$, which implies that $\rho_{j+\frac{1}{2}}^{n,-} = \rho_{j-\frac{1}{2}}^{n,+} = \rho_j^n$ for all $j \in \mathbb{Z}$. Using this, along with Remark 3.2.1, we further obtain $\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} = \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+} = \rho_j^n$ for all $j \in \mathbb{Z}$. Consequently, the scheme (3.2.5) with numerical flux (3.2.13) reduces to a first-order Lax-Friedrichs type scheme:

$$\rho_j^{n+1} = \rho_j^n - \lambda \left[F(\rho_j^n, \rho_{j+1}^n, A_{j+\frac{1}{2}}^n) - F(\rho_{j-1}^n, \rho_j^n, A_{j-\frac{1}{2}}^n) \right], \quad j \in \mathbb{Z}, \tag{3.2.15}$$

where $A_{j+\frac{1}{2}}^n := \dfrac{\Delta x}{2} \sum_{j \in \mathbb{Z}} (\mu_{j+1-l} + \mu_{j-l}) \rho_l^n \approx A(t^n, x_{j+\frac{1}{2}})$.

3.3 Estimates on the numerical solutions

In this section, we establish certain essential estimates on the approximate solutions generated by the scheme (3.2.5), namely L^∞, BV and L^1 time continuity estimates, which are required for the convergence analysis. Before proceeding into that, we start with some preliminary structural properties of the scheme. In the following remark, we obtain an estimate on the difference between the two consecutive left/right interface density values in (3.2.3).

Remark 3.3.1. From (3.2.2), we have $\sigma_j^n \sigma_{j+1}^n \geq 0$ for $j \in \mathbb{Z}$. This in turn implies that $|\sigma_{j+1}^n - \sigma_j^n| \leq \max\{|\sigma_{j+1}^n|, |\sigma_j^n|\}$. As a result, we obtain

$$\left| \frac{\sigma_{j+1}^n - \sigma_j^n}{\rho_{j+1}^n - \rho_j^n} \right| \leq \frac{\max\{|\sigma_{j+1}^n|, |\sigma_j^n|\}}{|\rho_{j+1}^n - \rho_j^n|} \leq 2\theta. \tag{3.3.1}$$

Consequently, using (3.2.3), for $\theta \in [0, 0.5]$, we write

$$\begin{aligned} \frac{\left(\rho_{j+\frac{3}{2}}^{n,-} - \rho_{j+\frac{1}{2}}^{n,-}\right)}{\rho_{j+1}^n - \rho_j^n} &= \frac{\rho_{j+1}^n - \rho_j^n}{\rho_{j+1}^n - \rho_j^n} + \frac{\left(\sigma_{j+1}^n - \sigma_j^n\right)}{2\left(\rho_{j+1}^n - \rho_j^n\right)} \geq 1 - \theta \geq \frac{1}{2}, \\ \frac{\left(\rho_{j+\frac{3}{2}}^{n,-} - \rho_{j+\frac{1}{2}}^{n,-}\right)}{\rho_{j+1}^n - \rho_j^n} &= \frac{\rho_{j+1}^n - \rho_j^n}{\rho_{j+1}^n - \rho_j^n} + \frac{\left(\sigma_{j+1}^n - \sigma_j^n\right)}{2\left(\rho_{j+1}^n - \rho_j^n\right)} \leq 1 + \theta \quad \text{for all}\, j \in \mathbb{Z}. \end{aligned} \tag{3.3.2}$$

Similarly, we obtain

$$\frac{1}{2} \leq \frac{\left(\rho_{j+\frac{1}{2}}^{n,+} - \rho_{j-\frac{1}{2}}^{n,+}\right)}{\rho_{j+1}^n - \rho_j^n} \leq 1 + \theta \quad \text{for all}\, j \in \mathbb{Z}. \tag{3.3.3}$$

Next, we present a bound on the reconstructed density values (3.2.3), in the following lemma.

Lemma 3.3.2. *(Bound on reconstructed values) Suppose that the piecewise constant approximate solution at the time level t^n given by the sequence $\{\rho_j^n\}_{j \in \mathbb{Z}}$, satisfies $\rho_j^n \geq 0$*

for all $j \in \mathbb{Z}$. Then, for each $j \in \mathbb{Z}$, the left and right interface values defined in (3.2.3) *are estimated as follows*

$$|\rho^{n,-}_{j+\frac{1}{2}}|, |\rho^{n,+}_{j-\frac{1}{2}}| \leq (1+\theta)\rho^n_j. \tag{3.3.4}$$

Proof. We split the proof into two cases:
Case (i) $(\sigma^n_j \geq 0)$: In this case, we observe that

$$\rho^{n,-}_{j+\frac{1}{2}} = \rho^n_j + \frac{1}{2}\sigma^n_j \geq \rho^n_j, \tag{3.3.5}$$

and

$$\rho^{n,-}_{j+\frac{1}{2}} = \rho^n_j + \frac{1}{2}\sigma_j \leq \rho^n_j + \frac{1}{2}2\theta(\rho^n_j - \rho^n_{j-1}) = (1+\theta)\rho^n_j - \theta\rho^n_{j-1} \leq (1+\theta)\rho^n_j. \tag{3.3.6}$$

Case (ii) $(\sigma^n_j < 0)$: In this case, we have

$$\rho^{n,-}_{j+\frac{1}{2}} = \rho^n_j + \frac{1}{2}\sigma_j \leq \rho^n_j, \tag{3.3.7}$$

and

$$\rho^{n,-}_{j+\frac{1}{2}} = \rho^n_j - 2\theta\frac{(\rho^n_j - \rho^n_{j+1})}{2} \geq (1-\theta)\rho^n_j + \theta\rho^n_{j+1} \geq 0. \tag{3.3.8}$$

By combining both the cases, it follows that $|\rho^{n,-}_{j+\frac{1}{2}}| \leq (1+\theta)\rho^n_j$ for all $j \in \mathbb{Z}$. In a similar way, we obtain $|\rho^{n,+}_{j-\frac{1}{2}}| \leq (1+\theta)\rho^n_j$ for all $j \in \mathbb{Z}$. □

Remark 3.3.3. For each $n \in \mathbb{N} \cup \{0\}$, the piecewise linear reconstruction $\tilde{\rho}^n_\Delta$ in (3.2.1) satisfies the following property on its total variation (see Lemma 3.1, Chapter 4, [100])

$$\mathrm{TV}(\tilde{\rho}^n_\Delta) \leq \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|. \tag{3.3.9}$$

Lemma 3.3.4. *(Bound on mid-time density values) Suppose that the piecewise constant approximate solution at the time level t^n given by the sequence $\{\rho^n_j\}_{j\in\mathbb{Z}}$, satisfies $\rho^n_j \geq 0$ for all $j \in \mathbb{Z}$. Then, the mid-time density values defined in* (3.2.6) *can be estimated as follows*

$$|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}|, |\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| \leq (1+\theta)(1+\lambda\|\partial_\rho f\|)\rho^n_j \quad \textit{for all}\, j \in \mathbb{Z}. \tag{3.3.10}$$

Proof. Using (3.2.6) and the hypothesis (H1), we estimate:

$$\begin{aligned}
|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}| &\leq |\rho^{n,-}_{j+\frac{1}{2}}| + \frac{\lambda}{2}|f(\rho^{n,-}_{j+\frac{1}{2}}, A^{n,-}_{j+\frac{1}{2}}) - f(0, A^{n,-}_{j+\frac{1}{2}}) + f(0, A^{n,+}_{j-\frac{1}{2}}) - f(\rho^{n,+}_{j-\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})| \\
&\leq |\rho^{n,-}_{j+\frac{1}{2}}| + \frac{\lambda}{2}|\partial_\rho f(\bar{\rho}^{n,-}_{j+\frac{1}{2}}, A^{n,-}_{j+\frac{1}{2}})||\rho^{n,-}_{j+\frac{1}{2}}| + \frac{\lambda}{2}|\partial_\rho f(\bar{\rho}^{n,+}_{j-\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})||\rho^{n,+}_{j-\frac{1}{2}}| \\
&\leq (1+\theta)(1+\lambda\|\partial_\rho f\|)\rho^n_j,
\end{aligned} \tag{3.3.11}$$

where $\bar{\rho}^{n,-}_{j+\frac{1}{2}} \in \mathcal{I}(0, \rho^{n,-}_{j+\frac{1}{2}})$ and $\bar{\rho}^{n,+}_{j-\frac{1}{2}} \in \mathcal{I}(0, \rho^{n,+}_{j-\frac{1}{2}})$. Similarly, we obtain $|\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| \leq (1+\theta)(1+\lambda\|\partial_\rho f\|)\rho^n_j$. □

Now, we establish that the proposed MUSCL-Hancock type scheme (3.2.5) gives non-negative solutions when the initial data is non-negative.

Theorem 3.3.5. *(Positivity-preserving property) Let the initial datum* $\rho_0 \in \mathrm{L}^\infty(\mathbb{R};\mathbb{R}_+)$*. Then, the approximate solutions* ρ_Δ *obtained from the proposed scheme* (3.2.5) *satisfies* $\rho_\Delta(t,x) \geq 0$ *for a.e.* $(t,x) \in \mathbb{R}\times\mathbb{R}_+$*, provided the CFL condition*

$$\frac{\Delta t}{\Delta x} \leq \min\left\{\frac{8-27\alpha}{27\|\partial_\rho f\|}, \frac{2}{27\|\partial_\rho f\|}, \frac{\alpha}{\|\partial_\rho f\|}\right\}, \tag{3.3.12}$$

holds.

Proof. To prove this result, we use the principle of mathematical induction. For the base case $n=0$, we have $\rho_j^0 \geq 0$ for all $j \in \mathbb{Z}$ by assumption. Now, for any $n \in \mathbb{N}\cup\{0\}$, assume that $\rho_j^n \geq 0$ for all $j\in\mathbb{Z}$. We show that $\rho_j^{n+1} \geq 0$ for all $j \in \mathbb{Z}$. To begin with, we write the scheme (3.2.5) in the form

$$\begin{aligned}\rho_j^{n+1} &= \rho_j^n - a_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\rho_j^n - \rho_{j-1}^n) + b_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\rho_{j+1}^n - \rho_j^n) \\ &\quad + \lambda\left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right],\end{aligned} \tag{3.3.13}$$

where we define

$$\begin{aligned}a_{j-\frac{1}{2}}^{n+\frac{1}{2}} &:= \lambda\frac{\left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})\right]}{(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-})}\left(\frac{\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}}{\rho_j^n - \rho_{j-1}^n}\right), \\ b_{j+\frac{1}{2}}^{n+\frac{1}{2}} &:= -\lambda\frac{\left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right]}{(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+})}\left(\frac{\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}}{\rho_{j+1}^n - \rho_j^n}\right).\end{aligned} \tag{3.3.14}$$

Now, using the definition (3.2.13), we observe that

$$\begin{aligned}&\frac{\left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})\right]}{(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-})} \\ &= \frac{1}{2}\frac{\left(f(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - f(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})\right)}{(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-})} + \frac{1}{2\lambda}\alpha \\ &= \frac{1}{2}\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \frac{1}{2\lambda}\alpha,\end{aligned} \tag{3.3.15}$$

for some $\bar{\rho}_j^{n+\frac{1}{2},-} \in \mathcal{I}(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-})$. Upon expanding the term $\dfrac{\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}}{\rho_j^n - \rho_{j-1}^n}$ using (3.2.6) and further using (3.3.15), we can write

$$a_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \frac{c_{j-\frac{1}{2}}^n}{2(\rho_j^n - \rho_{j-1}^n)}\Big(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\Big), \tag{3.3.16}$$

where we define

$$
\begin{aligned}
\bar{a}^{n+\frac{1}{2}}_{j-\frac{1}{2}} &:= \frac{1}{2}\left(\frac{\rho^{n,-}_{j+\frac{1}{2}} - \rho^{n,-}_{j-\frac{1}{2}}}{\rho^n_j - \rho^n_{j-1}}\right)\left(\lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_j, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) + \alpha\right), \\
c^n_{j-\frac{1}{2}} &:= -\frac{\lambda}{2}\left(f(\rho^{n,-}_{j+\frac{1}{2}}, A^{n,-}_{j+\frac{1}{2}}) - f(\rho^{n,+}_{j-\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})\right) \\
&\quad + \frac{\lambda}{2}\left(f(\rho^{n,-}_{j-\frac{1}{2}}, A^{n,-}_{j-\frac{1}{2}}) - f(\rho^{n,+}_{j-\frac{3}{2}}, A^{n,+}_{j-\frac{3}{2}})\right).
\end{aligned}
\tag{3.3.17}
$$

Now, proceeding as in (3.3.11) to apply the mean value theorem and subsequently employing the estimate (3.3.10), the term $c^n_{j-\frac{1}{2}}$ can be estimated as

$$
|c^n_{j-\frac{1}{2}}| \le \lambda\|\partial_\rho f\|(1+\theta)(\rho^n_j + \rho^n_{j-1}). \tag{3.3.18}
$$

Through similar arguments, the term $b^{n+\frac{1}{2}}_{j+\frac{1}{2}}$ in (3.3.14) also can be written as

$$
b^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \bar{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}} + \frac{1}{2}\frac{d^n_{j+\frac{1}{2}}}{(\rho^n_{j+1} - \rho^n_j)}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right), \tag{3.3.19}
$$

where $\bar{\rho}^{n+\frac{1}{2},+}_j \in \mathcal{I}(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}})$ and we define

$$
\begin{aligned}
\bar{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}} &:= \frac{1}{2}\left(\frac{\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}}{\rho^n_{j+1} - \rho^n_j}\right)\left(\alpha - \lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right), \\
d^n_{j+\frac{1}{2}} &:= -\frac{\lambda}{2}\left(f(\rho^{n,-}_{j+\frac{3}{2}}, A^{n,-}_{j+\frac{3}{2}}) - f(\rho^{n,+}_{j+\frac{1}{2}}, A^{n,+}_{j+\frac{1}{2}})\right) \\
&\quad + \frac{\lambda}{2}\left(f(\rho^{n,-}_{j+\frac{1}{2}}, A^{n,-}_{j+\frac{1}{2}}) - f(\rho^{n,+}_{j-\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})\right).
\end{aligned}
\tag{3.3.20}
$$

Furthermore, similar to (3.3.18), the term $d^n_{j+\frac{1}{2}}$ can be estimated as

$$
|d^n_{j+\frac{1}{2}}| \le \lambda\|\partial_\rho f\|(1+\theta)(\rho^n_j + \rho^n_{j+1}). \tag{3.3.21}
$$

In view of (3.3.16) and (3.3.19), the expression (3.3.13) now reduces to

$$
\begin{aligned}
\rho^{n+1}_j &= \rho^n_j(1 - \bar{a}^{n+\frac{1}{2}}_{j-\frac{1}{2}} - \bar{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}}) + \bar{a}^{n+\frac{1}{2}}_{j-\frac{1}{2}}\rho^n_{j-1} + \bar{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}}\rho^n_{j+1} \\
&\quad - \frac{c^n_{j-\frac{1}{2}}}{2}\left(\lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_j, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) + \alpha\right) + \frac{d^n_{j+\frac{1}{2}}}{2}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right) \\
&\quad + \lambda\left[F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right].
\end{aligned}
$$

As a consequence of (3.3.18) and (3.3.21) and the CFL condition (3.3.12) ($\lambda\|\partial_\rho f\| \le \alpha$), we subsequently obtain

$$
\begin{aligned}
\rho_j^{n+1} \geq & \Big(1 - \bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \bar{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\Big(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\Big) \\
& -\frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\Big(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\Big)\Big)\, \rho_j^n \\
& + \left(\bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right)\right)\rho_{j-1}^n \\
& + \left(\bar{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right)\right)\rho_{j+1}^n \\
& - \lambda\left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right].
\end{aligned}
\tag{3.3.22}
$$

Next, invoking Remark 3.3.1 (equation (3.3.2)), the CFL condition (3.3.12) and the fact that $\theta \in [0, 0.5]$, we obtain an estimate

$$
\begin{aligned}
& \bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right) \\
& = \frac{1}{2}\left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right)\left(\left(\frac{\rho_{j+\frac{1}{2}}^{n,-} - \rho_{j-\frac{1}{2}}^{n,-}}{\rho_j^n - \rho_{j-1}^n}\right) - \lambda\|\partial_\rho f\|(1+\theta)\right) \\
& \geq 0.
\end{aligned}
\tag{3.3.23}
$$

Similarly, we deduce that

$$
\bar{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right) \geq 0. \tag{3.3.24}
$$

Furthermore, using the definition of $\bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ from (3.3.17) and subsequently applying Remark 3.3.1 (equation (3.3.2)), we obtain

$$
\begin{aligned}
& \bar{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \frac{1}{2}\lambda\|\partial_\rho f\|(1+\theta)\left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right) \\
& = \frac{1}{2}\left(\left(\frac{\rho_{j+\frac{1}{2}}^{n,-} - \rho_{j-\frac{1}{2}}^{n,-}}{\rho_j^n - \rho_{j-1}^n}\right) + \lambda\|\partial_\rho f\|(1+\theta)\right)\left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right) \\
& \leq \frac{1}{2}(1+\theta)\left(1 + \lambda\|\partial_\rho f\|\right)\left(\lambda\|\partial_\rho f\| + \alpha\right) \leq \frac{3}{4} \times \frac{29}{27} \times \frac{8}{27} \leq \frac{1}{3},
\end{aligned}
\tag{3.3.25}
$$

where the last inequality is obtained from the CFL assumption (3.3.12). Analogously, we obtain

$$
\bar{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{1}{2}\lambda\|\partial_\rho f\|\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right) \leq \frac{1}{3}. \tag{3.3.26}
$$

Now, simplifying the last term in (3.3.22), using hypothesis (H1) along similar lines to (3.3.11), and subsequently using (3.3.10) and the CFL condition (3.3.12), we obtain the estimate

$$
\begin{aligned}
&\lambda|F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})-F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})|\\
&\quad=\frac{\lambda}{2}|f(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})+f(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})-f(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})-f(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})|\\
&\quad\leq\frac{\lambda}{2}\Big(|\partial_\rho f(\tilde{\rho}_{j+\frac{1}{2}}^{n+\frac{1}{2},-},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})||\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}|+|\partial_\rho f(\tilde{\rho}_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})||\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}|\\
&\qquad+|\partial_\rho f(\bar{\rho}_{j+\frac{1}{2}}^{n+\frac{1}{2},-},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})||\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}|+|\partial_\rho f(\hat{\rho}_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})||\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}|\Big)\\
&\quad\leq 2(1+\theta)\lambda\|\partial_\rho f\|(1+\lambda\|\partial_\rho f\|)\rho_j^n\leq 2\times\frac{3}{2}\times\frac{2}{27}\times\frac{29}{27}\rho_j^n\leq\frac{1}{3}\rho_j^n.
\end{aligned}
\tag{3.3.27}
$$

Finally, invoking the estimates (3.3.23), (3.3.24), (3.3.25), (3.3.26) and (3.3.27) in (3.3.22), we arrive at

$$
\rho_j^{n+1}\geq 0. \tag{3.3.28}
$$

This concludes the proof. □

Theorem 3.3.6. *(L^1-stability) Let the initial datum $\rho_0\in\mathrm{L}^\infty(\mathbb{R};\mathbb{R}_+)$. If the CFL condition* (3.3.12) *holds, then the approximate solution ρ_Δ computed using the proposed scheme* (3.2.5) *is L^1-stable:*

$$
\|\rho_\Delta(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}=\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\quad \textit{for all } t>0. \tag{3.3.29}
$$

Proof. From the definition of the numerical scheme (3.2.5) and using the positivity-preserving property (Theorem 3.3.5), for $t\in[t^n,t^{n+1})$, $n\in\mathbb{N}\cup\{0\}$, we write

$$
\begin{aligned}
\|\rho_\Delta(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}=\|\rho_\Delta(t^n,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}&=\Delta x\sum_{j\in\mathbb{Z}}|\rho_j^n|\\
&=\Delta x\sum_{j\in\mathbb{Z}}\rho_j^n\\
&=\Delta x\sum_{j\in\mathbb{Z}}\rho_j^{n-1}-\Delta t\sum_{j\in\mathbb{Z}}F_{j+\frac{1}{2}}^{n-\frac{1}{2}}+\Delta t\sum_{j\in\mathbb{Z}}F_{j-\frac{1}{2}}^{n-\frac{1}{2}}\\
&=\Delta x\sum_{j\in\mathbb{Z}}\rho_j^{n-1}=\cdots=\Delta x\sum_{j\in\mathbb{Z}}\rho_0^n=\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}.
\end{aligned}
\tag{3.3.30}
$$

□

Theorem 3.3.7. *(L^∞- stability) Let the initial datum $\rho_0\in\mathrm{L}^\infty(\mathbb{R};\mathbb{R}_+)$. If the CFL condition* (3.3.12) *holds, then there exists a constant C such that the approximate solutions ρ_Δ computed using the scheme* (3.2.5) *satisfy the $\mathrm{L}^\infty-$ estimate*

$$
\|\rho_\Delta(t,\cdot)\|\leq C, \tag{3.3.31}
$$

for all $t\in[0,T]$.

Proof. Recall the formulation (3.3.13) of the scheme (3.2.5):

$$\begin{aligned} \rho_j^{n+1} &= \rho_j^n - a_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\rho_j^n - \rho_{j-1}^n) + b_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\rho_{j+1}^n - \rho_j^n) \\ &\quad + \lambda \left[F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}}) \right]. \end{aligned} \tag{3.3.32}$$

First, using the definition (3.2.6), adding and subtracting suitable terms and applying the mean value theorem, we write

$$\begin{aligned} \frac{\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},-}}{\rho_j^n - \rho_{j-1}^n} &= \frac{\rho_{j+\frac{1}{2}}^{n,-} - \rho_{j-\frac{1}{2}}^{n,-}}{\rho_j^n - \rho_{j-1}^n} \left(1 - \frac{\lambda}{2}\partial_\rho f(\bar{\rho}_j^{n,-}, A_{j+\frac{1}{2}}^{n,-})\right) \\ &\quad + \frac{\rho_{j-\frac{1}{2}}^{n,+} - \rho_{j-\frac{3}{2}}^{n,+}}{\rho_j^n - \rho_{j-1}^n} \left(\frac{\lambda}{2}\partial_\rho f(\bar{\rho}_{j-1}^{n,+}, A_{j-\frac{1}{2}}^{n,+})\right) \\ &\quad - \frac{\lambda}{2(\rho_j^n - \rho_{j-1}^n)} \partial_A f(\rho_{j-\frac{1}{2}}^{n,-}, \bar{A}_j^{n,-}) \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right) \\ &\quad + \frac{\lambda}{2(\rho_j^n - \rho_{j-1}^n)} \partial_A f(\rho_{j-\frac{3}{2}}^{n,+}, \bar{A}_{j-1}^{n,+}) \left(A_{j-\frac{1}{2}}^{n,+} - A_{j-\frac{3}{2}}^{n,+}\right), \end{aligned} \tag{3.3.33}$$

where $\bar{\rho}_j^{n,-} \in \mathcal{I}(\rho_{j-\frac{1}{2}}^{n,-}, \rho_{j+\frac{1}{2}}^{n,-})$, $\bar{\rho}_{j-1}^{n,+} \in \mathcal{I}(\rho_{j-\frac{3}{2}}^{n,+}, \rho_{j-\frac{1}{2}}^{n,+})$, $\bar{A}_j^{n,-} \in \mathcal{I}(A_{j-\frac{1}{2}}^{n,-}, A_{j+\frac{1}{2}}^{n,-})$ and $\bar{A}_{j-1}^{n,+} \in \mathcal{I}(A_{j-\frac{3}{2}}^{n,+}, A_{j-\frac{1}{2}}^{n,+})$, for $j \in \mathbb{Z}$. Now using (3.3.33), the term $a_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ in (3.3.14) can be expressed as

$$a_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \frac{1}{2}\frac{\tilde{c}_{j-\frac{1}{2}}^n}{(\rho_j^n - \rho_{j-1}^n)} \left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right), \tag{3.3.34}$$

where

$$\begin{aligned} \tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} &:= \frac{1}{2}\hat{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left(\lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},-}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + \alpha\right), \\ \tilde{c}_{j-\frac{1}{2}}^n &:= -\frac{\lambda}{2}\left(\partial_A f(\rho_{j-\frac{1}{2}}^{n,-}, \bar{A}_j^{n,-}) \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right) - \partial_A f(\rho_{j-\frac{3}{2}}^{n,+}, \bar{A}_{j-1}^{n,+}) \left(A_{j-\frac{1}{2}}^{n,+} - A_{j-\frac{3}{2}}^{n,+}\right)\right), \\ \hat{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} &:= \left(\frac{\rho_{j+\frac{1}{2}}^{n,-} - \rho_{j-\frac{1}{2}}^{n,-}}{\rho_j^n - \rho_{j-1}^n}\right) \left(1 - \frac{\lambda}{2}\partial_\rho f(\bar{\rho}_j^{n,-}, A_{j+\frac{1}{2}}^{n,-})\right) + \left(\frac{\rho_{j-\frac{1}{2}}^{n,+} - \rho_{j-\frac{3}{2}}^{n,+}}{\rho_j^n - \rho_{j-1}^n}\right) \frac{\lambda}{2}\partial_\rho f(\bar{\rho}_{j-1}^{n,+}, A_{j-\frac{1}{2}}^{n,+}). \end{aligned}$$

Next, we show that the terms $\tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ satisfy

$$0 \le \tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} \le \frac{1}{2}, \quad \text{for all}\, j \in \mathbb{Z}. \tag{3.3.35}$$

First, using the estimate (3.3.2), the CFL condition (3.3.12) and the assumption that $\theta \in [0, 0.5]$, we have

$$\begin{aligned} \hat{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} &\ge \frac{1}{2}\left(1 - \frac{\lambda}{2}\|\partial_\rho f\|\right) - \frac{\lambda}{2}\|\partial_\rho f\|(1+\theta) \\ &= \frac{1}{2} - \lambda\|\partial_\rho f\| \left(\frac{1}{4} + \frac{1+\theta}{2}\right) \ge 0. \end{aligned} \tag{3.3.36}$$

The estimate (3.3.36) together with the CFL condition (3.3.12) yields $\tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} \geq 0$. Further, using (3.3.2) and the CFL condition (3.3.12), we derive an upper bound

$$\begin{aligned}\tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}} &\leq \frac{1}{2}(1+\theta)\left(1+\lambda\|\partial_\rho f\|\right)\left(\lambda\|\partial_\rho f\|+\alpha\right)\\ &\leq \frac{1}{2}\times\frac{3}{2}\times\frac{29}{27}\times\frac{8}{27}\leq\frac{1}{2},\end{aligned} \tag{3.3.37}$$

thereby verifying (3.3.37). Next, Theorems 3.3.5 and 3.3.6 imply that

$$|A_j^n - A_{j-1}^n| \leq \Delta x \sum_{l\in\mathbb{Z}} |\mu_{j-l} - \mu_{j-1-l}|\rho_l^n \leq \Delta x\|\mu'\|\|\rho_\Delta(t^n,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq \Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}. \tag{3.3.38}$$

Hence, from (3.2.11), we obtain

$$|s_j^n - s_{j-1}^n| \leq 2\theta\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}. \tag{3.3.39}$$

As a result, we arrive at

$$\begin{aligned}|A_{j-\frac{1}{2}}^{n,+} - A_{j-\frac{3}{2}}^{n,+}|, |A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}| &\leq |A_j^n - A_{j-1}^n| + \frac{1}{2}|s_j^n - s_{j-1}^n|\\ &\leq (1+\theta)\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})},\end{aligned} \tag{3.3.40}$$

which together with hypothesis (H2) and the estimate (3.3.4) yields

$$|\tilde{c}_{j-\frac{1}{2}}^n| \leq \lambda M(1+\theta)^2\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\rho_{j-1}^n. \tag{3.3.41}$$

In an analogous way, the term $b_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ in (3.3.14) can be expressed as

$$b_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \tilde{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{1}{2}\frac{\tilde{d}_{j+\frac{1}{2}}^n}{(\rho_{j+1}^n - \rho_j^n)}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right), \tag{3.3.42}$$

where

$$\begin{aligned}\tilde{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} &:= \frac{1}{2}\hat{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right),\\ \tilde{d}_{j+\frac{1}{2}}^n &:= -\frac{\lambda}{2}\Big(\partial_A f(\rho_{j+\frac{1}{2}}^{n,-}, \bar{A}_{j+1}^{n,-})\left(A_{j+\frac{3}{2}}^{n,-} - A_{j+\frac{1}{2}}^{n,-}\right)\\ &\qquad - \partial_A f(\rho_{j-\frac{1}{2}}^{n,+}, \bar{A}_j^{n,+})\left(A_{j+\frac{1}{2}}^{n,+} - A_{j-\frac{1}{2}}^{n,+}\right)\Big),\\ \hat{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} &:= \left(\frac{\rho_{j+\frac{1}{2}}^{n,+} - \rho_{j-\frac{1}{2}}^{n,+}}{\rho_{j+1}^n - \rho_j^n}\right)(1 + \frac{\lambda}{2}\partial_\rho f(\bar{\rho}_j^{n,+}, A_{j+\frac{1}{2}}^{n,+}))\\ &\qquad - \left(\frac{\rho_{j+\frac{3}{2}}^{n,-} - \rho_{j+\frac{1}{2}}^{n,-}}{\rho_{j+1}^n - \rho_j^n}\right)\frac{\lambda}{2}\partial_\rho f(\bar{\rho}_{j+1}^{n,-}, A_{j+\frac{3}{2}}^{n,-}),\end{aligned} \tag{3.3.43}$$

where $\bar{\rho}_j^{n,-} \in \mathcal{I}(\rho_{j-\frac{1}{2}}^{n,-}, \rho_{j+\frac{1}{2}}^{n,-}), \bar{\rho}_{j-1}^{n,+} \in \mathcal{I}(\rho_{j-\frac{3}{2}}^{n,+}, \rho_{j-\frac{1}{2}}^{n,+}), \bar{A}_j^{n,-} \in \mathcal{I}(A_{j-\frac{1}{2}}^{n,-}, A_{j+\frac{1}{2}}^{n,-})$ and $\bar{A}_{j-1}^{n,+} \in \mathcal{I}(A_{j-\frac{3}{2}}^{n,+}, A_{j-\frac{1}{2}}^{n,+})$, for $j\in\mathbb{Z}$. Further, we obtain the following bounds:

$$0 \leq \tilde{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \leq \frac{1}{2}, \tag{3.3.44}$$

and

$$|\tilde{d}^n_{j+\frac{1}{2}}| \leq \lambda M(1+\theta)^2 \Delta x \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \rho^n_j. \tag{3.3.45}$$

Using (3.3.34) and (3.3.42), the scheme (3.3.32) now reads as

$$\begin{aligned} \rho^{n+1}_j &= \rho^n_j (1 - \tilde{a}^{n+\frac{1}{2}}_{j-\frac{1}{2}} - \tilde{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}}) + \tilde{a}^{n+\frac{1}{2}}_{j-\frac{1}{2}} \rho^n_{j-1} + \tilde{b}^{n+\frac{1}{2}}_{j+\frac{1}{2}} \rho^n_{j+1} \\ &\quad - \frac{1}{2} \tilde{c}^n_{j-\frac{1}{2}} \left(\lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_j, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) + \alpha \right) \\ &\quad + \frac{1}{2} \tilde{d}^n_{j+\frac{1}{2}} \left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) \right) \\ &\quad + \lambda \left[F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) \right]. \end{aligned} \tag{3.3.46}$$

Lemma 3.3.4 (equation (3.3.10)) and Theorems 3.3.5 and 3.3.6 together yield

$$\begin{aligned} |A^{n+\frac{1}{2}}_{j-\frac{1}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}}| &\leq \frac{\Delta x}{2} \sum_{l\in\mathbb{Z}} |\mu_{j+1-l} - \mu_{j-l}| |\rho^{n+\frac{1}{2},+}_{l-\frac{1}{2}}| \\ &\quad + \frac{\Delta x}{2} \sum_{l\in\mathbb{Z}} |\mu_{j-l} - \mu_{j-l-1}| |\rho^{n+\frac{1}{2},-}_{l+\frac{1}{2}}| \\ &\leq \frac{\Delta x^2}{2} \|\mu'\| \sum_{l\in\mathbb{Z}} (|\rho^{n+\frac{1}{2},+}_{l-\frac{1}{2}}| + |\rho^{n+\frac{1}{2},-}_{l+\frac{1}{2}}|) \\ &\leq \Delta x \|\mu'\| (1+\theta)(1+\lambda\|\partial_\rho f\|) \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}. \end{aligned} \tag{3.3.47}$$

Furthermore, using the mean value theorem, hypothesis (H2) and the estimates (3.3.47) and (3.3.10), we obtain

$$\begin{aligned} &\lambda \big| F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) \big| \\ &= \frac{\lambda}{2} \left| \left(\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_j) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \hat{A}^{n+\frac{1}{2}}_j) \right) (A^{n+\frac{1}{2}}_{j-\frac{1}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) \right| \\ &\leq \frac{\lambda}{2} M \left(|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}| + |\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| \right) |A^{n+\frac{1}{2}}_{j-\frac{1}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}}| \\ &\leq \Delta t M (1+\theta)^2 (1+\lambda\|\partial_\rho f\|)^2 \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \|\rho_\Delta(t^n,\cdot)\|. \end{aligned} \tag{3.3.48}$$

Finally, combining the estimates (3.3.35), (3.3.44), (3.3.41), (3.3.45) and (3.3.48) in (3.3.46), we arrive at

$$\begin{aligned} |\rho^{n+1}_j| &\leq \|\rho_\Delta(t^n,\cdot)\| + \Delta t(\alpha + \lambda\|\partial_\rho f\|) M(1+\theta)^2 \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \|\rho_\Delta(t^n,\cdot)\| \\ &\quad + \Delta t M(1+\theta)^2 (1+\lambda\|\partial_\rho f\|)^2 \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \|\rho_\Delta(t^n,\cdot)\| \\ &\leq \|\rho_\Delta(t^n,\cdot)\| (1+\tilde{\mathcal{C}}\Delta t) \\ &\leq (1+\tilde{\mathcal{C}}\Delta t)^{n+1} \|\rho_0\| \leq \exp(\tilde{\mathcal{C}}(n+1)\Delta t) \|\rho_0\|, \end{aligned} \tag{3.3.49}$$

where $\tilde{\mathcal{C}} := [\alpha + \lambda\|\partial_\rho f\| + (1+\lambda\|\partial_\rho f\|)^2] M(1+\theta)^2 \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}$. Now, choosing $C := \exp(\tilde{\mathcal{C}}T)\|\rho_0\|$, the result (3.3.31) follows. □

Theorem 3.3.8. *(Total variation estimate) If the initial datum* $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$ *and the CFL condition* (3.3.12) *holds, then the approximate solutions* ρ_Δ *computed using the scheme* (3.2.5) *satisfy*

$$\mathrm{TV}(\rho_\Delta(t,\cdot)) \leq \exp(\mathcal{A}T)\left(\sum_{j\in\mathbb{Z}}|\rho^0_{j+1}-\rho^0_j|\right) + \mathcal{B}(\exp(\mathcal{A}T)-1), \tag{3.3.50}$$

for all $t \in [0,T]$ *and for some constants* $\mathcal{A},\mathcal{B} > 0$.

Proof. From the scheme (3.2.5), computing the difference $\rho^{n+1}_{j+1}-\rho^{n+1}_j$ and subsequently adding and subtracting suitable terms, we obtain

$$\rho^{n+1}_{j+1}-\rho^{n+1}_j = C^n_{j+\frac{1}{2}} - \lambda D^n_{j+\frac{1}{2}}, \tag{3.3.51}$$

where

$$\begin{aligned} C^n_{j+\frac{1}{2}} := (\rho^n_{j+1}-\rho^n_j) &- \lambda\left(F(\rho^{n+\frac{1}{2},-}_{j+\frac{3}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{3}{2}},A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{3}{2}})\right) \\ &+ \lambda\left(F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right), \end{aligned} \tag{3.3.52}$$

and

$$\begin{aligned} D^n_{j+\frac{1}{2}} := &\left(F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right) \\ &- \left(F(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}},A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right). \end{aligned} \tag{3.3.53}$$

Now, we expand the term $C^n_{j+\frac{1}{2}}$ in (3.3.52) by adding and subtracting appropriate terms as follows

$$C^n_{j+\frac{1}{2}} = (1-\ell^n_{j+\frac{1}{2}}-k^n_{j+\frac{1}{2}})(\rho^n_{j+1}-\rho^n_j) + \ell^n_{j+\frac{3}{2}}(\rho^n_{j+2}-\rho^n_{j+1}) + k^n_{j-\frac{1}{2}}(\rho^n_j-\rho^n_{j-1}), \tag{3.3.54}$$

where

$$\begin{aligned} k^n_{j+\frac{1}{2}} &:= \lambda\left(\frac{F(\rho^{n+\frac{1}{2},-}_{j+\frac{3}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{3}{2}})}{\rho^{n+\frac{1}{2},-}_{j+\frac{3}{2}}-\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}}\right)\left(\frac{\rho^{n+\frac{1}{2},-}_{j+\frac{3}{2}}-\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}}{\rho^n_{j+1}-\rho^n_j}\right), \\ \ell^n_{j+\frac{1}{2}} &:= -\lambda\left(\frac{F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - F(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}},\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}},A^{n+\frac{1}{2}}_{j+\frac{1}{2}})}{\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}-\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}}\right)\left(\frac{\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}-\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}}{\rho^n_{j+1}-\rho^n_j}\right). \end{aligned} \tag{3.3.55}$$

Starting from (3.2.6), we consider the difference $\rho^{n+\frac{1}{2},-}_{j+\frac{3}{2}}-\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}$. Then we rearrange the terms, add and subtract suitable terms and finally apply the mean value theorem to obtain

$$
\begin{aligned}
\rho^{n+\frac12,-}_{j+\frac32} - \rho^{n+\frac12,-}_{j+\frac12} &= \left(\rho^{n,-}_{j+\frac32} - \rho^{n,-}_{j+\frac12}\right) - \frac{\lambda}{2}\left(f(\rho^{n,-}_{j+\frac32}, A^{n,-}_{j+\frac32}) - f(\rho^{n,-}_{j+\frac12}, A^{n,-}_{j+\frac12})\right) \\
&\quad + \frac{\lambda}{2}\left(f(\rho^{n,+}_{j+\frac12}, A^{n,+}_{j+\frac12}) - f(\rho^{n,+}_{j-\frac12}, A^{n,+}_{j-\frac12})\right) \\
&= \left(\rho^{n,-}_{j+\frac32} - \rho^{n,-}_{j+\frac12}\right)\left(1 - \frac{\lambda}{2}\partial_\rho f(\bar\rho^{n,-}_{j+1}, A^{n,-}_{j+\frac32})\right) \\
&\quad + \left(\rho^{n,+}_{j+\frac12} - \rho^{n,+}_{j-\frac12}\right)\left(\frac{\lambda}{2}\partial_\rho f(\bar\rho^{n,+}_{j}, A^{n,+}_{j+\frac12})\right) \\
&\quad - \frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac12}, \bar A^{n,-}_{j+1})\left(A^{n,-}_{j+\frac32} - A^{n,-}_{j+\frac12}\right) \\
&\quad + \frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac12}, \bar A^{n,+}_{j})\left(A^{n,+}_{j+\frac12} - A^{n,+}_{j-\frac12}\right),
\end{aligned}
\tag{3.3.56}
$$

where $\bar\rho^{n,-}_j \in \mathcal{I}(\rho^{n,-}_{j-\frac12},\ \rho^{n,-}_{j+\frac12}), \bar\rho^{n,+}_{j-1} \in \mathcal{I}(\rho^{n,+}_{j-\frac32},\ \rho^{n,+}_{j-\frac12}), \bar A^{n,-}_j \in \mathcal{I}(A^{n,-}_{j-\frac12}, A^{n,-}_{j+\frac12})$ and $\bar A^{n,+}_{j-1} \in \mathcal{I}(A^{n,+}_{j-\frac32}, A^{n,+}_{j-\frac12})$, for $j \in \mathbb{Z}$. Noting that

$$
\lambda\left(\frac{F(\rho^{n+\frac12,-}_{j+\frac32}, \rho^{n+\frac12,+}_{j+\frac12}, A^{n+\frac12}_{j+\frac32}) - F(\rho^{n+\frac12,-}_{j+\frac12}, \rho^{n+\frac12,+}_{j+\frac12}, A^{n+\frac12}_{j+\frac32})}{\rho^{n+\frac12,-}_{j+\frac32} - \rho^{n+\frac12,-}_{j+\frac12}}\right) = \frac12\left(\lambda\partial_\rho f(\bar\rho^{n+\frac12,-}_{j+1}, A^{n+\frac12}_{j+\frac32}) + \alpha\right),
$$

where $\bar\rho^{n+\frac12,-}_{j+1} \in \mathcal{I}(\rho^{n+\frac12,-}_{j+\frac12}, \rho^{n+\frac12,-}_{j+\frac32})$ and in view of (3.3.56), we rewrite the term $k^n_{j+\frac12}$ in (3.3.55) as

$$
k^n_{j+\frac12} = \tilde k^n_{j+\frac12} + \frac{\hat k^n_{j+\frac12}}{\rho^n_{j+1} - \rho^n_j}, \tag{3.3.57}
$$

where

$$
\begin{aligned}
\tilde k^n_{j+\frac12} &:= \frac12\left(\lambda\partial_\rho f(\bar\rho^{n+\frac12,-}_{j+1}, A^{n+\frac12}_{j+\frac32}) + \alpha\right)\left(\left(\frac{\rho^{n,-}_{j+\frac32} - \rho^{n,-}_{j+\frac12}}{\rho^n_{j+1} - \rho^n_j}\right)\left(1 - \frac{\lambda}{2}\partial_\rho f(\bar\rho^{n,-}_{j+1}, A^{n,-}_{j+\frac32})\right)\right. \\
&\qquad \left. + \left(\frac{\rho^{n,+}_{j+\frac12} - \rho^{n,+}_{j-\frac12}}{\rho^n_{j+1} - \rho^n_j}\right)\left(\frac{\lambda}{2}\partial_\rho f(\bar\rho^{n,+}_{j}, A^{n,+}_{j+\frac12})\right)\right), \\
\hat k^n_{j+\frac12} &:= \frac12\left(\lambda\partial_\rho f(\bar\rho^{n+\frac12,-}_{j+1}, A^{n+\frac12}_{j+\frac32}) + \alpha\right)\left(-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac12}, \bar A^{n,-}_{j+1})\left(A^{n,-}_{j+\frac32} - A^{n,-}_{j+\frac12}\right)\right. \\
&\qquad \left. + \frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac12}, \bar A^{n,+}_{j})\left(A^{n,+}_{j+\frac12} - A^{n,+}_{j-\frac12}\right)\right).
\end{aligned}
\tag{3.3.58}
$$

Analogously, we can write $\ell^n_{j+\frac12}$ (3.3.55) as

$$
\ell^n_{j+\frac12} = \tilde\ell^n_{j+\frac12} + \frac{\hat\ell^n_{j+\frac12}}{\rho^n_{j+1} - \rho^n_j}, \tag{3.3.59}
$$

where

$$
\begin{aligned}
\tilde{\ell}^n_{j+\frac{1}{2}} &:= \frac{1}{2}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left(\left(\frac{\rho^{n,-}_{j+\frac{3}{2}} - \rho^{n,-}_{j+\frac{1}{2}}}{\rho^n_{j+1} - \rho^n_j}\right)\left(-\frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,-}_{j+1}, A^{n,-}_{j+\frac{3}{2}})\right)\right.\\
&\quad \left. + \left(\frac{\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}}{\rho^n_{j+1} - \rho^n_j}\right)\left(1 + \frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,+}_j, A^{n,+}_{j+\frac{1}{2}})\right)\right),\\
\hat{\ell}^n_{j+\frac{1}{2}} &:= \frac{1}{2}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left(-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_{j+1})\left(A^{n,-}_{j+\frac{3}{2}} - A^{n,-}_{j+\frac{1}{2}}\right)\right.\\
&\quad \left. + \frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac{1}{2}}, \bar{A}^{n,+}_j)\left(A^{n,+}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}}\right)\right).
\end{aligned}
\tag{3.3.60}
$$

In the light of (3.3.57) and (3.3.59), the term $C^n_{j+\frac{1}{2}}$ in (3.3.52) can be expressed as

$$
C^n_{j+\frac{1}{2}} = \tilde{C}^n_{j+\frac{1}{2}} + \hat{C}^n_{j+\frac{1}{2}}, \tag{3.3.61}
$$

where

$$
\begin{aligned}
\tilde{C}^n_{j+\frac{1}{2}} &:= (1 - \tilde{\ell}^n_{j+\frac{1}{2}} - \tilde{k}^n_{j+\frac{1}{2}})(\rho^n_{j+1} - \rho^n_j) + \tilde{\ell}^n_{j+\frac{3}{2}}(\rho^n_{j+2} - \rho^n_{j+1}) + \tilde{k}^n_{j-\frac{1}{2}}(\rho^n_j - \rho^n_{j-1}),\\
\hat{C}^n_{j+\frac{1}{2}} &:= -\hat{\ell}^n_{j+\frac{1}{2}} - \hat{k}^n_{j+\frac{1}{2}} + \hat{\ell}^n_{j+\frac{3}{2}} + \hat{k}^n_{j-\frac{1}{2}}.
\end{aligned}
\tag{3.3.62}
$$

From (3.3.51) and (3.3.61), it follows that

$$
\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \le \sum_{j\in\mathbb{Z}} |\tilde{C}^n_{j+\frac{1}{2}}| + \sum_{j\in\mathbb{Z}} |\hat{C}^n_{j+\frac{1}{2}}| + \lambda\sum_{j\in\mathbb{Z}} |D^n_{j+\frac{1}{2}}|. \tag{3.3.63}
$$

The terms on the right-hand side of (3.3.63) are estimated (derived in Appendix B.1) as follows

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |\tilde{C}^n_{j+\frac{1}{2}}| &\le \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|,\\
\sum_{j\in\mathbb{Z}} |\hat{C}^n_{j+\frac{1}{2}}| &\le 4\mathcal{K}_1\Delta t + 4\mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|,\\
\lambda\sum_{j\in\mathbb{Z}} |D^n_{j+\frac{1}{2}}| &\le \Delta t\mathcal{K}_7 + \Delta t\mathcal{K}_8\sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|,
\end{aligned}
\tag{3.3.64}
$$

where $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_7, \mathcal{K}_8$ are constants independent of Δx defined in (B.1.23) and (B.1.41). Now, invoking the estimates (3.3.64) in (3.3.63) yields

$$
\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \le \Delta t\mathcal{K}_9 + (1 + \mathcal{K}_{10}\Delta t)\sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|, \tag{3.3.65}
$$

where

$$
\begin{aligned}
\mathcal{K}_9 &:= 4\mathcal{K}_1 + \mathcal{K}_7,\\
\mathcal{K}_{10} &:= 4\mathcal{K}_2 + \mathcal{K}_8.
\end{aligned}
$$

Starting from (3.3.65) and proceeding recursively, we obtain

$$\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n+1}-\rho_j^{n+1}|\leq(1+\mathcal{K}_{10}\Delta t)^{n+1}\left(\sum_{j\in\mathbb{Z}}|\rho_{j+1}^0-\rho_j^0|\right)+\mathcal{N}(n), \tag{3.3.66}$$

where $\mathcal{N}(n):=\mathcal{K}_9\Delta t\,(1+(1+\mathcal{K}_{10}\Delta t)+\cdots+(1+\mathcal{K}_{10}\Delta t)^{n-1}+(1+\mathcal{K}_{10}\Delta t)^n)$.

Noting that $(1+\mathcal{K}_{10}\Delta t)^{n+1}\leq\exp(\mathcal{K}_{10}T)$ for $(n+1)\Delta t\leq T$, the term $\mathcal{N}(n)$ in (3.3.66) can be simplified as

$$\begin{aligned}\mathcal{N}(n)&=\frac{\mathcal{K}_9}{\mathcal{K}_{10}}\left((1+\mathcal{K}_{10}\Delta t)-1\right)\left[1+(1+\mathcal{K}_{10}\Delta t)+\cdots+(1+\mathcal{K}_{10}\Delta t)^{n-1}+(1+\mathcal{K}_{10}\Delta t)^n\right]\\&=\frac{\mathcal{K}_9}{\mathcal{K}_{10}}\left[(1+\mathcal{K}_{10}\Delta t)+\cdots+(1+\mathcal{K}_{10}\Delta t)^n+(1+\mathcal{K}_{10}\Delta t)^{n+1}\right.\\&\qquad\qquad\left.-1-(1+\mathcal{K}_{10}\Delta t)\cdots-(1+\mathcal{K}_{10}\Delta t)^n\right]\\&=\frac{\mathcal{K}_9}{\mathcal{K}_{10}}\left((1+\mathcal{K}_{10}\Delta t)^{n+1}-1\right)\leq\frac{\mathcal{K}_9}{\mathcal{K}_{10}}(\exp(\mathcal{K}_{10}T)-1).\end{aligned} \tag{3.3.67}$$

Finally, in light of (3.3.67) from (3.3.66), we deduce

$$\mathrm{TV}(\rho_\Delta^{n+1})=\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n+1}-\rho_j^{n+1}|\leq\exp(\mathcal{K}_{10}T)\left(\sum_{j\in\mathbb{Z}}|\rho_{j+1}^0-\rho_j^0|\right)+\frac{\mathcal{K}_9}{\mathcal{K}_{10}}(\exp(\mathcal{K}_{10}T)-1), \tag{3.3.68}$$

for $(n+1)\Delta t\leq T$. Now, upon choosing $\mathcal{A}:=\mathcal{K}_{10}$ and $\mathcal{B}:=\dfrac{\mathcal{K}_9}{\mathcal{K}_{10}}$, the result (3.3.50) is immediate. □

Theorem 3.3.9. *(L^1-Lipschitz continuity in time) Let the initial datum $\rho_0\in\mathrm{L}^\infty\cap\mathrm{BV}(\mathbb{R};\mathbb{R}_+)$. If the CFL condition* (3.3.12) *holds, then there exists a constant κ such that the approximate solutions generated by the proposed scheme* (3.2.5) *satisfy*

$$\|\rho_\Delta(t_1,\cdot)-\rho_\Delta(t_2,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}\leq\kappa(|t_1-t_2|+\Delta t), \tag{3.3.69}$$

for any $t_1,t_2\in[0,T]$.

Proof. Recalling the scheme (3.2.5) written in the form (3.3.46):

$$\begin{aligned}\rho_j^{n+1}=\;&\rho_j^n(1-\tilde a_{j-\frac12}^{n+\frac12}-\tilde b_{j+\frac12}^{n+\frac12})+\tilde a_{j-\frac12}^{n+\frac12}\rho_{j-1}^n+\tilde b_{j+\frac12}^{n+\frac12}\rho_{j+1}^n\\&-\frac12\tilde c_{j-\frac12}^n\left(\lambda\partial_\rho f(\bar\rho_j^{n+\frac12,-},A_{j-\frac12}^{n+\frac12})+\alpha\right)+\frac12\tilde d_{j+\frac12}^n\left(\alpha-\lambda\partial_\rho f(\bar\rho_j^{n+\frac12,+},A_{j+\frac12}^{n+\frac12})\right)\\&+\lambda\left[F(\rho_{j+\frac12}^{n+\frac12,-},\rho_{j-\frac12}^{n+\frac12,+},A_{j-\frac12}^{n+\frac12})-F(\rho_{j+\frac12}^{n+\frac12,-},\rho_{j-\frac12}^{n+\frac12,+},A_{j+\frac12}^{n+\frac12})\right],\end{aligned}$$

we write

$$
\begin{aligned}
&\|\rho_\Delta(t^{n+1},\cdot)-\rho_\Delta(t^n,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}\\
&\quad=\Delta x\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n+1}-\rho_j^n|\\
&\quad\leq \Delta x\sum_{j\in\mathbb{Z}}|\rho_{j-1}^n-\rho_j^n||\tilde{a}_{j-\frac{1}{2}}^{n+\frac{1}{2}}|+\Delta x\sum_{j\in\mathbb{Z}}|\rho_{j+1}^n-\rho_j^n||\tilde{b}_{j+\frac{1}{2}}^{n+\frac{1}{2}}|\\
&\qquad+\frac{1}{2}\Delta x\left(\lambda\|\partial_\rho f\|+\alpha\right)\sum_{j\in\mathbb{Z}}\left(|\tilde{c}_{j-\frac{1}{2}}^n|+|\tilde{d}_{j+\frac{1}{2}}^n|\right)\\
&\qquad+\Delta t\sum_{j\in\mathbb{Z}}\Big|F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})-F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\Big|.
\end{aligned}
\tag{3.3.70}
$$

In order to estimate the last term in (3.3.70), we apply the mean value theorem and subsequently use the estimate (3.3.47), hypothesis (H2) and Lemma 3.3.4 to yield

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}|F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j-\frac{1}{2}}^{n+\frac{1}{2}})-F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},A_{j+\frac{1}{2}}^{n+\frac{1}{2}})|\\
&\quad\leq\frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\left|\partial_A f(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-},\bar{A}_j^{n+\frac{1}{2}})\right|+\left|\partial_A f(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+},\tilde{A}_j^{n+\frac{1}{2}})\right|\right)|A_{j-\frac{1}{2}}^{n+\frac{1}{2}}-A_{j+\frac{1}{2}}^{n+\frac{1}{2}}|\\
&\quad\leq M\|\mu'\|(1+\theta)^2(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\Delta x\sum_{j\in\mathbb{Z}}\rho_j^n\\
&\quad\leq M\|\mu'\|(1+\theta)^2(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}^2.
\end{aligned}
\tag{3.3.71}
$$

Now, invoking the estimates (3.3.35), (3.3.44), (3.3.41), (3.3.45) (derived in the proof of Theorem 3.3.7) and (3.3.71), the L^1 distance in (3.3.70) can be estimated as

$$
\begin{aligned}
\|\rho_\Delta(t^{n+1},\cdot)-\rho_\Delta(t^n,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}&\leq\Delta t\left(\frac{1}{\lambda}\sum_{j\in\mathbb{Z}}|\rho_{j+1}^n-\rho_j^n|+\mathcal{J}\right)\\
&\leq\Delta t\kappa,
\end{aligned}
\tag{3.3.72}
$$

for $(n+1)\Delta t\leq T$, where

$$
\begin{aligned}
\mathcal{J}&:=M\|\mu'\|(1+\theta)^2\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}^2(\alpha+\lambda\|\partial_\rho f\|+(1+\lambda\|\partial_\rho f\|)^2),\\
\kappa&:=\frac{1}{\lambda}\left(\exp(\mathcal{A}T)\mathrm{TV}(\rho_0)+\mathcal{B}(\exp(\mathcal{A}T)-1)\right)+\mathcal{J},
\end{aligned}
$$

and the last inequality follows from Theorem 3.3.8. The result (3.3.69) is now an immediate consequence of (3.3.72).

□

3.4 Convergence of the numerical scheme

In this section, we show that the numerical scheme (3.2.5) converges to the unique entropy solution of the problem (3.1.1). To begin with, we recall a result originally established in [166], and adapted to the case of non-local conservation laws in Chapter 2 (see Theorem 2.4.1).

Theorem 3.4.1. *Suppose that a numerical scheme that approximates* (3.1.1) *can be written in the form:*

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1} - e_{j+\frac{1}{2}}^{n+1} + e_{j-\frac{1}{2}}^{n+1}, \tag{3.4.1}$$

where
(i) $\tilde{\rho}_j^{n+1}$ is computed from ρ_j^n using a scheme which yields a sequence of approximate solutions converging in $\mathrm{L}^1_{\mathrm{loc}}$ to the entropy solution of (3.1.1).
(ii) $|e_{j+\frac{1}{2}}^{n+1}| \leq K\Delta x^\delta$ for $(n+1)\Delta t \leq T$, where $K > 0$ and $\delta \in (0,1)$ are some constants independent of Δx.
(iii) The approximate solutions ρ_Δ obtained using (2.4.1) *are in L^∞, BV, and admits L^1-Lipschitz continuity in time.*
Then the approximate solutions generated by the scheme (2.4.1) converges in $\mathrm{L}^1_{\mathrm{loc}}$ to the entropy solution of (3.1.1).

The proof of this result follows along similar lines to that of Theorem 2.4.1 from Chapter 2. Now our goal is to write the scheme (3.2.5) in the form (2.4.1) and show that it satisfies all the hypotheses of Theorem 3.4.1. To this end, we first observe that the scheme (3.2.5) can be written in the form

$$\begin{aligned} \tilde{\rho}_j^{n+1} &= \rho_j^n - \lambda\left[F(\rho_j^n, \rho_{j+1}^n, A_{j+\frac{1}{2}}^n) - F(\rho_{j-1}^n, \rho_j^n, A_{j-\frac{1}{2}}^n)\right], \\ \rho_j^{n+1} &= \tilde{\rho}_j^{n+1} - e_{j+\frac{1}{2}}^{n+1} + e_{j-\frac{1}{2}}^{n+1}, \end{aligned} \tag{3.4.2}$$

where $A_{j+\frac{1}{2}}^n$ is as defined in (3.2.15) and

$$e_{j+\frac{1}{2}}^{n+1} := \lambda\left(F(\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},-}, \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - F(\rho_j^n, \rho_{j+1}^n, A_{j+\frac{1}{2}}^n)\right). \tag{3.4.3}$$

Additionally, we modify the scheme (3.4.2) by redefining the slopes (3.2.2) as follows

$$\sigma_j^n = 2\theta \mathrm{minmod}\left((\rho_j^n - \rho_{j-1}^n),\ \frac{1}{2}(\rho_{j+1}^n - \rho_{j-1}^n),\ (\rho_{j+1}^n - \rho_j^n), \mathrm{sgn}(\rho_{j+1}^n - \rho_j^n)\mathcal{K}(\Delta x)^\delta\right), \tag{3.4.4}$$

for some $\mathcal{K} > 0$ and some $\delta \in (0,1)$.

Lemma 3.4.2. *There exists a constant $K > 0$ such that the correction terms $\{e_{j+\frac{1}{2}}^{n+1}\}_{j\in\mathbb{Z}}$ in the scheme* (3.4.2) *with the modified slopes* (3.4.4) *satisfy*

$$|e_{j+\frac{1}{2}}^{n+1}| \leq K(\Delta x)^\delta \quad \textit{for } j \in \mathbb{Z} \textit{ and } (n+1)\Delta t \leq T. \tag{3.4.5}$$

Proof. We expand $e^{n+1}_{j+\frac{1}{2}}$, add and subtract suitable terms and subsequently use the mean value theorem to write

$$\begin{aligned}
e^{n+1}_{j+\frac{1}{2}} &= \frac{\lambda}{2}\left(\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^n_j) + \partial_A f(\rho^n_j, \bar{A}^n_{j+\frac{1}{2}})(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^n_{j+\frac{1}{2}})\right)\\
&\quad + \frac{\lambda}{2}\left(\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^n_{j+1}) + \partial_A f(\rho^n_{j+1}, \tilde{A}^n_{j+\frac{1}{2}})(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^n_{j+\frac{1}{2}})\right)\\
&\quad - \frac{1}{2}\alpha\left(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^n_{j+1}\right) + \frac{1}{2}\alpha\left(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^n_j\right)\\
&= \left(\frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) + \frac{1}{2}\alpha\right)(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^n_j)\\
&\quad + \left(\frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - \frac{1}{2}\alpha\right)(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^n_{j+1})\\
&\quad + \left(\partial_A f(\rho^n_j, \bar{A}^n_{j+\frac{1}{2}}) + \partial_A f(\rho^n_{j+1}, \tilde{A}^n_{j+\frac{1}{2}})\right)(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^n_{j+\frac{1}{2}}),
\end{aligned} \tag{3.4.6}$$

where $\bar{\rho}^{n+\frac{1}{2},-}_{j+\frac{1}{2}} \in \mathcal{I}(\rho^n_j, \rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}})$, $\bar{\rho}^{n+\frac{1}{2},+}_{j+\frac{1}{2}} \in \mathcal{I}(\rho^n_{j+1}, \rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}})$, and $\bar{A}^n_{j+\frac{1}{2}}, \tilde{A}^n_{j+\frac{1}{2}} \in \mathcal{I}(A^n_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})$. In order to obtain a bound on $e^{n+1}_{j+\frac{1}{2}}$, it is sufficient to estimate the terms $|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^n_j|$, $|\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}} - \rho^n_j|$ and $|A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^n_{j+\frac{1}{2}}|$. To this end, we first note that

$$|A^{n,-}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}}| = |s^n_j| = \theta|A^n_{j+1} - A^n_{j-1}| \leq \theta\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1\mathbb{R}}. \tag{3.4.7}$$

Now, expanding $\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}$ and $\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}$ from (3.2.6), adding and subtracting appropriate terms, subsequently applying the mean value theorem and hypothesis (H2), and finally applying Lemma 3.3.2, Theorem 3.3.7 and the estimate (3.4.7), we obtain

$$\begin{aligned}
|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^n_j| &\leq \frac{1}{2}|\sigma^n_j| + \frac{\lambda}{2}\big|(f(\rho^{n,-}_{j+\frac{1}{2}}, A^{n,-}_{j+\frac{1}{2}}) - f(\rho^{n,+}_{j-\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})\big|\\
&\leq \frac{1}{2}|\sigma^n_j| + \frac{\lambda}{2}\big|(\partial_\rho f(\bar{\rho}^n_j, A^{n,-}_{j+\frac{1}{2}})(\rho^{n,-}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}})\big|\\
&\quad + \frac{\lambda}{2}\big|\partial_A f(\rho^{n,+}_{j-\frac{1}{2}}, \bar{A}^n_j)(A^{n,-}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}})\big|\\
&\leq \left(\frac{1}{2} + \frac{\lambda}{2}\|\partial_\rho f\|\right)|\sigma^n_j| + \frac{\lambda}{2}M|\rho^{n,+}_{j-\frac{1}{2}}|\big|A^{n,-}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}}\big|\\
&\leq \left(\frac{1}{2} + \frac{\lambda}{2}\|\partial_\rho f\|\right)\mathcal{K}(\Delta x)^\delta + \left(\frac{\lambda}{2}MC(1+\theta)\theta\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\right)\Delta x\\
&\leq \tilde{\mathcal{K}}(\Delta x)^\delta,
\end{aligned} \tag{3.4.8}$$

where $\bar{A}^n_j \in \mathcal{I}(A^{n,-}_{j+\frac{1}{2}}, A^{n,+}_{j-\frac{1}{2}})$ and $\bar{\rho}^n_j \in \mathcal{I}(\rho^{n,-}_{j+\frac{1}{2}}, \rho^{n,+}_{j-\frac{1}{2}})$ and $\tilde{\mathcal{K}} := \left(\frac{1}{2} + \frac{\lambda}{2}\|\partial_\rho f\|\right)\mathcal{K} + \frac{\lambda}{2}MC(1+\theta)\theta\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}$. Identically, we obtain

$$|\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}} - \rho^n_j| \leq \tilde{\mathcal{K}}(\Delta x)^\delta. \tag{3.4.9}$$

Further, we write

$$\begin{aligned}|A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n}_{j+\frac{1}{2}}| &\leq \frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}\left[|\mu_{j+1-l}||\rho^{n+\frac{1}{2},+}_{l-\frac{1}{2}} - \rho^n_l| + |\mu_{j-l}||\rho^{n+\frac{1}{2},-}_{l+\frac{1}{2}} - \rho^n_l|\right] \\ &\leq \tilde{\mathcal{K}}(\Delta x)^{\delta}\|\mu\|L_\mu,\end{aligned} \tag{3.4.10}$$

where L_μ denotes the length of a compact interval which contains the support of the convolution kernel μ. Now, invoking the estimates (3.4.8), (3.4.9) and (3.4.10) in (3.4.3), we obtain the desired result (3.4.5) with $K = (\lambda\|\partial_\rho f\| + \alpha + 2M\|\mu\|L_\mu)\,\tilde{\mathcal{K}}$. $\square$

With the necessary estimates at hand, we are now in a position to use Theorem 3.4.3 to establish the convergence of the numerical scheme to the entropy solution. We present this result in the following theorem.

Theorem 3.4.3. *Let* $\rho_0 \in \mathrm{L}^\infty \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$. *If the CFL condition* (3.3.12) *holds, then the approximate solutions* ρ_Δ *generated by the scheme* (3.2.5) *with the modified slopes* (3.4.4)*, converge to the unique entropy solution of the problem* (3.1.1)*.*

Proof. It is sufficient to verify that the scheme (3.2.5), with modified slopes (3.4.4), satisfies the hypotheses of Theorem 3.4.1. The formulation (3.4.2) shows that the scheme can be expressed in the form (2.4.1). Further, $\tilde{\rho}^{n+1}_j$ in (3.4.2) is computed using the first-order Lax-Friedrichs scheme (3.2.15), whose convergence to the entropy solution has been established in [33, 19, 14]. This verifies hypothesis (i) of Theorem 3.4.1. Hypothesis (ii) holds true due to the result in Lemma 3.4.2. Finally, we observe that the results in Theorems 3.3.7, 3.3.8 and 3.3.9 hold true for the scheme (3.2.5) together with the modified slopes (3.4.4), as well. This confirms the validity of condition (iii). Hence, the result is proved. $\square$

Remark 3.4.4. While implementing the scheme, the slope modification (3.4.4) is not really needed. This is because, for any given mesh-size Δx, we can choose a sufficiently large constant $\mathcal{K} > 0$, so that the modified slope (3.4.4) reduces to (3.2.2). Specifically, for mesh sizes $\Delta x \geq \epsilon$ for some fixed $\epsilon > 0$, we can choose $\mathcal{K} = 2C\epsilon^{-\delta}$, where C is as in Theorem 3.3.7. This has also been observed in [131] (p. 158), [166] (p. 68) and [165] (p. 577).

Remark 3.4.5. Note that the slope modification (3.4.4) is introduced solely to ensure the entropy convergence of the scheme. Even without the modification (3.4.4), the scheme (3.2.5) can be independently shown to converge to a weak solution (3.1.3) of the problem (3.1.1): Theorems 3.3.7, 3.3.8 and 3.3.9 allow us to apply Kolmogorov's compactness theorem (see Theorem 2.3.1, Chapter 2), which guarantees the existence of a subsequence $\Delta_k \to 0$ and a function $\rho \in C([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ such that ρ_{Δ_k} converges to ρ in $C([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$. Further, using a Lax-Wendroff-type argument (see the proof of Theorem 2.3.2, Chapter 2), we can show that ρ is a weak solution of the problem (3.1.1).

3.5 Numerical experiments

In this section, we present numerical experiments to illustrate the performance of the proposed scheme (3.2.5), in comparison with the first-order Lax–Friedrichs (FO) scheme (3.2.15). To further demonstrate the advantage of this particular second-order MH-type scheme, we also compare it with a standard second-order MUSCL-Runge-Kutta (RK-2) scheme (see Section B.2). For all schemes, we choose the time step to satisfy the CFL condition (3.3.12) corresponding to the MH scheme, with the coefficient α set to 0.16. We consider a uniform discretization of the spatial domain $I = [x_l, x_r]$ into M cells of size $\Delta x = (x_r - x_l)/M$, and denote the cell averages at time t^n by $\{\rho_j^n\}_{j=1}^M$. Let $[a, b]$ be the smallest compact interval containing the support of the measure μ, and define $N := (b - a)/\Delta x$.

To implement the boundary conditions, we introduce ghost cells on either side of the domain by defining the values $\rho_{-N+1}^n, \rho_{-N+2}^n, \ldots, \rho_0^n$, and $\rho_{M+1}^n, \rho_{M+2}^n, \ldots, \rho_{M+N}^n$, on the left and right of the domain, respectively. We consider two types of boundary conditions:

1. Periodic boundary conditions:

$$\begin{aligned} \rho_{M+i}^n &:= \rho_i^n, && \text{for } i = 1, \ldots, N, \\ \rho_{-i}^n &:= \rho_{M-i+1}^n, && \text{for } i = 0, \ldots, N-1. \end{aligned}$$

2. Absorbing boundary conditions:

$$\begin{aligned} \rho_{M+i}^n &:= \rho_M^n, && \text{for } i = 1, \ldots, N, \\ \rho_{-i}^n &:= \rho_1^n, && \text{for } i = 0, \ldots, N-1. \end{aligned}$$

Example 3.1.(Smooth solution case) As studied in [19], we consider an example of the problem (3.1.1), where the flux function is given by

$$f(\rho, A) = \rho(1 - \rho)(1 - A) \tag{3.5.1}$$

and the convolution kernel is chosen as

$$\mu(x) := \frac{1}{\mathcal{M}} \left((x - a)(b - x)\right)^{\frac{5}{2}} \chi_{[a,b]}(x), \quad x \in \mathbb{R}, \tag{3.5.2}$$

where $\mathcal{M} := \int_a^b \left((x - a)(b - x)\right)^{\frac{5}{2}} \mathrm{d}x$ and $[a, b] \subset \mathbb{R}$. To evaluate the experimental order of accuracy, we consider the smooth initial condition:

$$\rho_0(x) = 0.5 + 0.4 \sin(\pi x) \tag{3.5.3}$$

in the domain $[-1,1]$ together with three sets of $[a,b]$ given by: $[0.0,0.25]$, $[-0.125,0.125]$ and $[-0.25,0.0]$ corresponding to the upstream, centered and downstream convolutions, respectively. We evolve the solutions imposing periodic boundary conditions up to time $t=0.15$, for mesh sizes $\Delta x \in \{0.2, 0.1, 0.05, 0.025, 0.0125\}$. The MH solution corresponding to the fine mesh size $\Delta x = 2/640$ is taken as the reference solution which we denote by ρ_{ref}. The experimental order of accuracy (E.O.A.) is then computed using the formula

$$\Theta(\Delta x) = \log_2 \left(\frac{\|\rho_{2\Delta x} - \rho_{ref}\|_{L^1}}{\|\rho_{\Delta x} - \rho_{ref}\|_{L^1}} \right). \tag{3.5.4}$$

The results presented in Table 3.1 show that the MH scheme attains the expected order of accuracy. Further, the corresponding L^1 error versus CPU time plots for the MH and RK-2 schemes are given in Figure 3.1. While Table 3.1 indicates that the the RK-2 scheme attains a similar E.O.A. to that of the MH scheme, the results in Figure 3.1 show that the MH scheme is computationally more efficient.

Kernel support $[a,b]$		FO		MH		RK-2	
	Δx	L^1- error	$\Theta(\Delta x)$	L^1- error	$\Theta(\Delta x)$	L^1- error	$\Theta(\Delta x)$
$[0.0, 0.25]$ (upstream convolution)	0.2	0.206012	-	0.082694	-	0.082462	-
	0.1	0.115288	0.837490	0.027777	1.573868	0.027355	1.591917
	0.05	0.063176	0.867773	0.008861	1.648333	0.008904	1.619287
	0.025	0.033041	0.935116	0.002471	1.842107	0.002530	1.815118
	0.0125	0.017026	0.956479	0.000644	1.939505	0.000657	1.943746
$[-0.125, 0.125]$ (centered convolution)	0.2	0.191868	-	0.071610	-	0.071878	-
	0.1	0.108584	0.821301	0.027806	1.364778	0.027328	1.395156
	0.05	0.058011	0.904406	0.008895	1.644184	0.008844	1.627536
	0.025	0.030147	0.944303	0.002527	1.815667	0.002544	1.797536
	0.0125	0.015438	0.965463	0.000655	1.946496	0.000661	1.944145
$[-0.25, 0.0]$ (downstream convolution)	0.2	0.179211	-	0.070700	-	0.070413	-
	0.1	0.102061	0.812219	0.026216	1.431257	0.025736	1.452034
	0.05	0.054620	0.901929	0.008712	1.589371	0.008626	1.576991
	0.025	0.028217	0.952864	0.002590	1.749549	0.002600	1.729973
	0.0125	0.014402	0.970304	0.000696	1.894568	0.000699	1.893973

Table 3.1: Example 3.1. L^1- errors and E.O.A. obtained for FO, MH and RK-2 schemes, with $\Delta t = \Delta x/20$, computed at time $t=0.15$.

Example 3.2. We consider the same setup as in Example 3.1 but with a discontinuous initial datum

$$\rho_0(x) = \frac{1}{2}\chi_{[-2.8,-1.8]}(x) + \frac{3}{4}\chi_{[-1.2,-0.2]}(x) + \frac{3}{4}\chi_{[0.6,1.0]}(x) + \chi_{[1.5,+\infty)}(x) \tag{3.5.5}$$

and $[a,b]=[-0.25,0.0]$ (downstream convolution). The numerical solutions are computed in the domain $[-3.0,3.0]$ up to time $t=2.5$ using absorbing boundary conditions and the

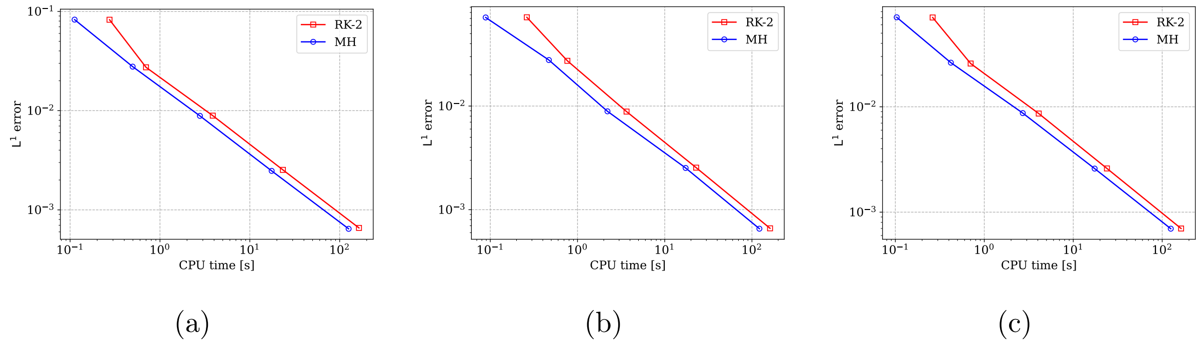


Figure 3.1: Example 3.1. Log–log plots of L^1 error versus CPU time for the MH and RK-2 schemes applied to (3.1.1) with the smooth initial condition (3.5.3) at time $t = 0.15$. Results are displayed for three different choices of the interval $[a, b]$ in (3.5.2): (a) $[0.0, 0.25]$ (upstream convolution), (b) $[-0.125, 0.125]$ (centered convolution), and (c) $[-0.25, 0.0]$ (downstream convolution).

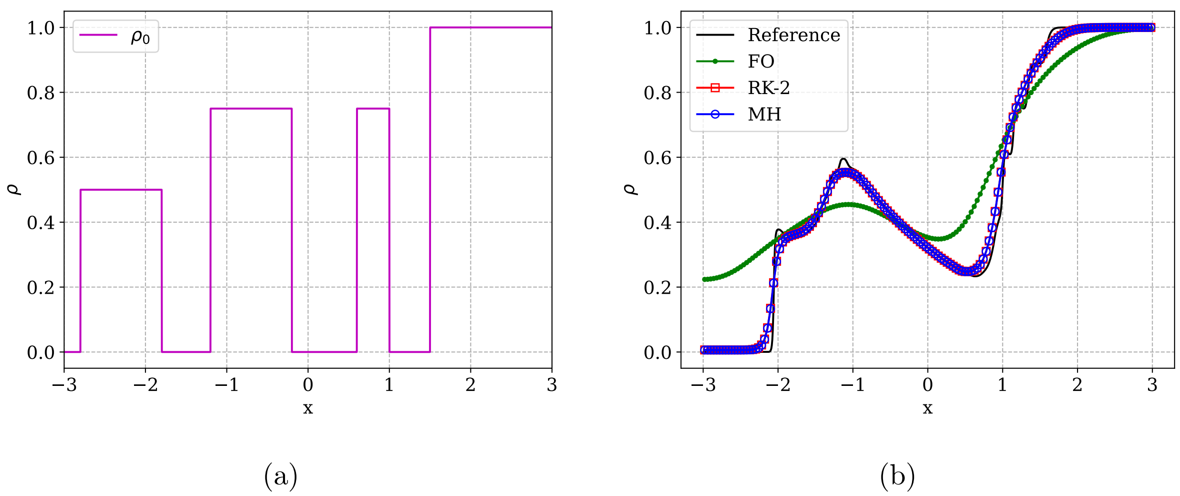
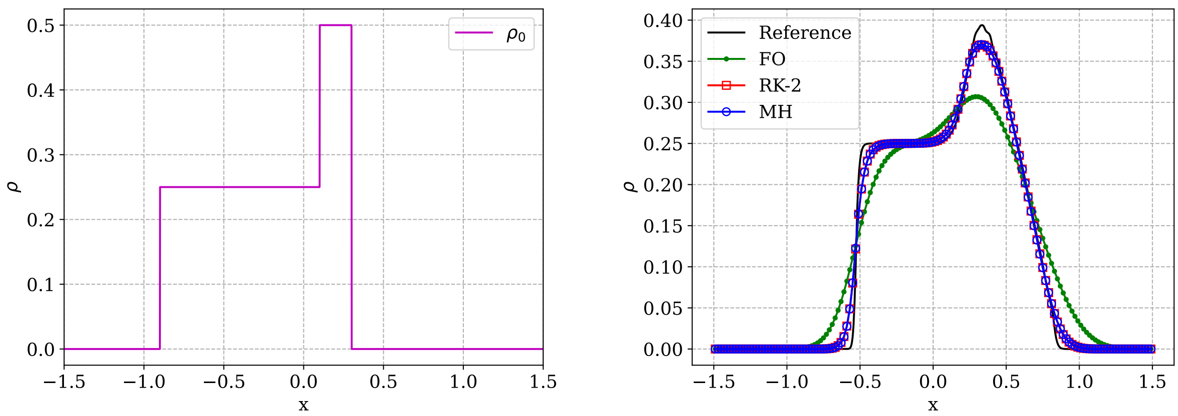


Figure 3.2: Example 3.2. (a) Initial datum given in (3.5.5). (b) Numerical solutions at time $t = 2.5$, computed with the kernel function (3.5.2) with $[a, b] = [-0.25, 0.0]$ using $\Delta x = 6/150$ and $\Delta t = \Delta x/20$.

results are displayed in Figure 3.2. Here, the reference solution is computed using the MH scheme with a fine mesh of size $\Delta x = 6/900$. While both the MH and RK-2 schemes provide a comparable resolution as seen in Figure 3.2, the L^1 error versus CPU time plots presented in Figure 3.4(a) , computed for mesh sizes $\Delta x \in \{6/40, 6/80, 6/160, 6/320\}$, indicate that the MH scheme is computationally more efficient.

Example 3.3. In this example, we consider the problem (3.1.1) with a different flux function given by

$$f(\rho, A) = \rho(1 - A), \tag{3.5.6}$$



(a) (b)

Figure 3.3: Example 3.3. (a) Initial datum given in (3.5.8). (b) Numerical solutions at time $t = 0.5$, computed with the kernel function (3.5.7), using $\Delta x = 3/150$ and $\Delta t = \Delta x/20$.

and the kernel function μ given by

$$\mu(x) := \frac{1}{\left(\int_{-\eta}^{0}(-x(\eta+x))^3\,\mathrm{d}x\right)}(-x(\eta+x))^3\chi_{[-\eta,0]}, \tag{3.5.7}$$

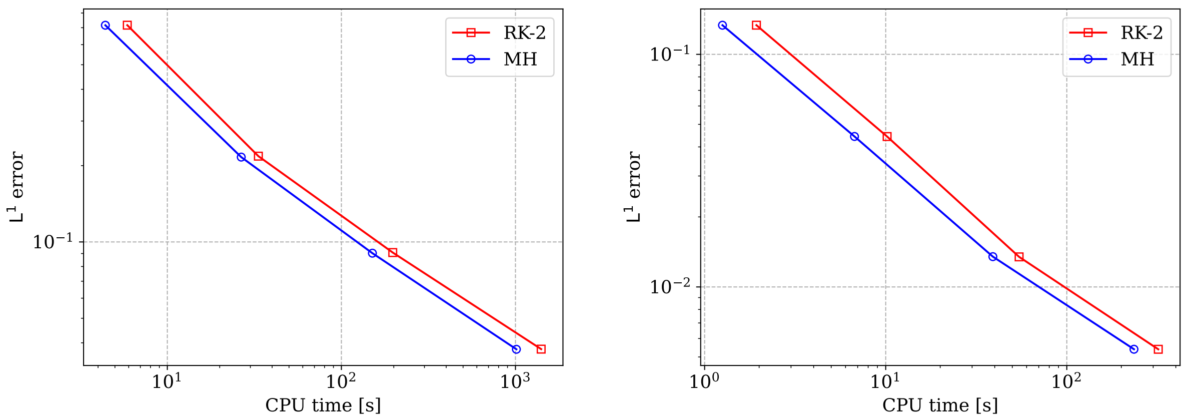
considered in [15]. We choose a discontinuous initial datum

$$\rho_0(x) = \begin{cases} 0.25 & \text{for } -0.9 \le x \le 0.1, \\ 0.5 & \text{for } 0.1 \le x \le 0.3 \end{cases} \tag{3.5.8}$$

and evolve the numerical solutions in the domain $[-1.5, 1.5]$ up to time $t = 0.5$ using absorbing boundary conditions and the results are illustrated in Figure 3.3. Here, the reference solution is computed using the MH scheme with a mesh of size $\Delta x = 3/900$. Additionally, we present the L^1 error versus CPU time plots in Figure 3.4(b), computed for mesh sizes $\Delta x \in \{6/40, 6/80, 6/160, 6/320\}$. The comparison in Figure 3.3 shows that the MH scheme (and RK-2) outperforms the FO scheme, as expected. As in Example 3.2, although the MH and RK-2 schemes produce comparable solutions, Figure 3.4(b) indicates that the MH scheme is computationally more efficient.

3.6 Concluding remarks

We have proposed a single-stage MUSCL–Hancock-type second-order scheme for a general class of non-local conservation laws and established its convergence analysis. The construction of the scheme relies on a careful design of the discrete convolutions, which is crucial not only for achieving second-order accuracy but also for enabling a rigorous convergence analysis. The numerical results presented in Section 3.5 confirm that the



(a) (b)

Figure 3.4: Examples 3.2 and 3.3. Log–log plots of L^1 error versus CPU time for the MH and RK-2 schemes for discontinuous solutions. (a) Example 3.2 and (b) Example 3.3.

proposed scheme produces numerical solution of significantly improved accuracy than a typical first-order method. To highlight the advantage of the proposed MH-type scheme, we also compare it with a standard second-order MUSCL–RK-2 method. For the test cases in Examples 3.2 and 3.3, both the MH and RK-2 schemes produce comparable solutions. However, the results shown in Figures 3.4(a) and 3.4(b) reveal that the MH scheme is computationally more efficient than the RK-2 scheme, underscoring the significance of the proposed method.

4

Second-order scheme for multidimensional non-local systems

In this chapter, we focus on the numerical discretization of a system of non-local conservation laws in several space dimensions (described in Chapter 1, Section 1.1). For the one-dimensional case, non-local conservation laws have been well-studied in the literature from both theoretical and numerical perspectives (see, e.g., [15, 16, 17, 98, 110, 119] and references therein). However, their extension to multiple space dimensions is comparatively less explored, with only a limited number of results addressing their well-posedness. For instance, the authors in [14] proved the existence of a weak solution for a general system in two dimensions by establishing the convergence of a dimensionally split scheme with Lax-Friedrichs numerical flux. Additionally, the existence and uniqueness of measure-valued solutions to a class of multi-dimensional problems are analyzed in [72]. Local-in-time existence and uniqueness results for certain multi-dimensional non-local equations under weak differentiability assumptions on the convolution kernel are recently studied in [66]. Furthermore, [15] presents the error analysis of first-order finite volume schemes for a one-dimensional problem, including a discussion of its extension to the multi-dimensional case. In this work, we are interested in the general system of multi-dimensional non-local conservation laws treated in [14].

We develop a second-order fully discrete numerical scheme for the class of multidimensional non-local systems considered in [14]. The scheme combines a MUSCL-type spatial reconstruction [162] with a second-order strong stability-preserving Runge–Kutta (SSP-RK) time-stepping method [103, 104]. As a key contribution of this work, we show that the resulting scheme satisfies the positivity-preserving property. This property is particularly important in models such as crowd dynamics, where the unknowns represent densities of different species and must remain non-negative. In addition, we establish that the numerical solutions obtained from the proposed second-order scheme are L^∞- stable. To evaluate the performance of the proposed scheme, we conduct several numerical experiments on two well-known non-local problems in two space dimensions: a crowd dynamics model [14] posed as a scalar non-local conservation law, and the Keyfitz–Kranzer system [122], a model arising in elasticity theory. The results collectively highlight the advantages of the proposed scheme over its first-order counterpart. Furthermore, the asymptotic compatibility of the proposed scheme is numerically investigated in the context of the singular limit problem as the non-local horizon parameter tends to zero (see [64, 65, 110]).

The rest of this chapter is organized as follows. In Section 4.1, we outline the non-local system of conservation laws under consideration. Section 4.2 presents the second-order numerical scheme. The positivity-preserving property of the scheme is established in Section 4.3, while Section 4.4 is devoted to proving the L^∞- stability. Numerical examples are provided in Section 4.5, and the conclusions are summarized in Section 4.6.

4.1 System of non-local conservation laws

We consider a system of non-local conservation laws in n space dimensions, previously studied in [14]:

$$\partial_t \boldsymbol{\rho} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\eta}_1 * \boldsymbol{\rho}, \cdots, \boldsymbol{\eta}_n * \boldsymbol{\rho}) = 0, \tag{4.1.1}$$

where $\boldsymbol{x} := (x_1, x_2, \cdots, x_n)$ and the unknown is

$$\boldsymbol{\rho} := \left(\rho^1, \rho^2, \cdots, \rho^N\right),$$

and for each fixed $r \in \{1, 2, \ldots, n\}$, the convolution kernel corresponding to the r-th dimension is given by the $m \times N$ matrix

$$\boldsymbol{\eta}_r := \begin{pmatrix} \eta_r^{1,1} & \cdots & \eta_r^{1,N} \\ \vdots & \ddots & \vdots \\ \eta_r^{m,1} & \cdots & \eta_r^{m,N} \end{pmatrix},$$

where $\eta_r^{l,k} : \mathbb{R}^n \to \mathbb{R}$. The system (4.1.1) is posed along with the initial condition

$$\boldsymbol{\rho}(\boldsymbol{x}, 0) = \boldsymbol{\rho}_0(\boldsymbol{x}). \tag{4.1.2}$$

For simplicity, we restrict our attention to the case of systems of non-local conservation laws in two dimensions, i.e., $n = 2$ and $\boldsymbol{x} = (x, y)$. Note that all the results in this case can be readly extended to the case of general n-dimensional systems. The convolution kernel functions corresponding to the x-and y-direction are then given by the matrices

$$\boldsymbol{\eta} := \boldsymbol{\eta}_1 = \begin{pmatrix} \eta^{1,1} & \cdots & \eta^{1,N} \\ \vdots & \ddots & \vdots \\ \eta^{m,1} & \cdots & \eta^{m,N} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu} := \boldsymbol{\eta}_2 = \begin{pmatrix} \nu^{1,1} & \cdots & \nu^{1,N} \\ \vdots & \ddots & \vdots \\ \nu^{m,1} & \cdots & \nu^{m,N} \end{pmatrix},$$

respectively, and the flux function takes the form

$$\boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\eta} * \boldsymbol{\rho}, \boldsymbol{\nu} * \boldsymbol{\rho}) := \begin{pmatrix} f^1(t,x,y,\rho^1, \boldsymbol{\eta} * \boldsymbol{\rho}) & g^1(t,x,y,\rho^1, \boldsymbol{\nu} * \boldsymbol{\rho}) \\ \vdots & \vdots \\ f^N(t,x,y,\rho^N, \boldsymbol{\eta} * \boldsymbol{\rho}) & g^N(t,x,y,\rho^N, \boldsymbol{\nu} * \boldsymbol{\rho}) \end{pmatrix}^T.$$

For $k \in \{1, \ldots, N\}$, we now focus on the problem associated with the k-th unknown ρ^k of the problem (4.1.1)-(4.1.2), given by

$$\begin{aligned} \partial_t \rho^k + \partial_x f^k + \partial_y g^k = 0, \ t > 0, \ (x,y) \in \mathbb{R}^2, \\ \boldsymbol{\rho}(0,x,y) = \left(\rho_0^k(x,y)\right)_{k=1}^N, \ (x,y) \in \mathbb{R}^2, \end{aligned} \tag{4.1.3}$$

where $f^k = f^k(t,x,y,\rho^k, \boldsymbol{\eta} * \boldsymbol{\rho})$ and $g^k = g^k(t,x,y,\rho^k, \boldsymbol{\nu} * \boldsymbol{\rho})$, and the convolution terms are defined as

$$\begin{aligned} \left(\boldsymbol{\eta} * \boldsymbol{\rho}\right)_q(t,x,y) &:= \sum_{k=1}^N \int\int_{\mathbb{R}^2} \eta^{q,k}(x-x', y-y') \rho^k(t,x',y') \, \mathrm{d}x' \, \mathrm{d}y', \\ \left(\boldsymbol{\nu} * \boldsymbol{\rho}\right)_q(t,x,y) &:= \sum_{k=1}^N \int\int_{\mathbb{R}^2} \nu^{q,k}(x-x', y-y') \rho^k(t,x',y') \, \mathrm{d}x' \, \mathrm{d}y', \end{aligned}$$

for $q \in \{1, 2, \ldots, m\}$.

4.1.1 Notations

Apart from the notations in Chapter 1, Section 1.0.0.1, in what follows, we denote $\mathbb{R}_+^N := [0, \infty)^N$. For a vector-valued quantity $\boldsymbol{\rho} : \mathbb{R}^2 \to \mathbb{R}^N$, we define $\|\boldsymbol{\rho}\| := \max_{k \in \{1,2,\ldots,N\}} \|\rho^k\|$ and $\|\boldsymbol{\rho}\|_{\mathrm{L}^1} := \sum_{k=1}^N \|\rho^k\|_{\mathrm{L}^1}$. Also, for a matrix-valued quantity $\boldsymbol{\eta} : \mathbb{R}^2 \to \mathbb{R}^{m \times N}$, we define $\|\partial_x \boldsymbol{\eta}\| := \max_{q,k} \|\partial_x \eta^{q,k}\|$ and $\|\partial_y \boldsymbol{\eta}\| := \max_{q,k} \|\partial_y \eta^{q,k}\|$. Further, for any vectors $\boldsymbol{A_1}, \boldsymbol{A_2} \in \mathbb{R}^m$, $\tilde{\mathcal{I}}(\boldsymbol{A_1}, \boldsymbol{A_2}) := \{\kappa \boldsymbol{A_1} + (1-\kappa)\boldsymbol{A_2} \mid \kappa \in (0,1)\}$.

4.1.2 Hypotheses

In this work, the non-local problem (4.1.1)-(4.1.2) is studied under the following hypotheses:
(H0) For all $t \in \mathbb{R}_+$, $(x,y) \in \mathbb{R}^2$ and $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^m$, for all $k \in \{1, 2, \ldots, N\}$ the flux functions satisfy

1. $f^k, g^k \in \mathrm{C}^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$.

2. $\partial_\rho f^k, \partial_\rho g^k \in \mathrm{L}^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$.

3. $f^k(t, x, y, 0, \boldsymbol{A}) = g^k(t, x, y, 0, \boldsymbol{B}) = 0$.

(H1) There exists a constant $M > 0$ such that for all $t, x, y, \rho, \boldsymbol{A}$ and $\boldsymbol{B}$ in the respective domains

$$|\partial_x f^k|, \|\nabla_A f^k\| \leq M|\rho| \quad \text{and} \quad |\partial_y g^k|, \|\nabla_B g^k\| \leq M|\rho| \quad \text{for}\, k \in \{1, 2, \ldots, N\}.$$

(H2) $\boldsymbol{\eta}, \boldsymbol{\nu} \in (\mathrm{C}^2 \cap \mathrm{W}^{1,\infty})(\mathbb{R}^2; \mathbb{R}^{m \times N})$.

We note that under these assumptions, along with some additional hypotheses, the existence of a weak solution to problem (4.1.1)-(4.1.2) was established in [14] through the convergence of a first-order numerical scheme employing a Lax-Friedrichs-type numerical flux (see Theorem 2.3, [14]).

4.2 Second-order scheme

We discretize the spatial domain into Cartesian grids of the form $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, where we define

$$x_i := i\Delta x,\ y_j := j\Delta y,\ x_{i+\frac{1}{2}} := (i + \frac{1}{2})\Delta x \text{ and } y_{j+\frac{1}{2}} := (j + \frac{1}{2})\Delta y,\ \forall\, i, j \in \mathbb{Z}.$$

The time domain is discretized using a time-step Δt and we denote $t^n = n\Delta t$ for $n \in \mathbb{N}$. We also denote $\lambda_x := \dfrac{\Delta t}{\Delta x}$ and $\lambda_y := \dfrac{\Delta t}{\Delta y}$. The initial datum $\boldsymbol{\rho}_0$ is discretized as

$$\rho_{ij}^{k,0} = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \rho_0^k(x, y)\, \mathrm{d}x\, \mathrm{d}y \quad \text{for } i, j \in \mathbb{Z}.$$

To construct a second-order scheme, we essentially follow the method-of-lines approach, in which we consider a semi-discrete finite volume formulation combined with a piecewise linear spatial reconstruction of the approximate solution. Time evolution is then performed using a strong stability preserving (SSP) second-order Runge–Kutta method. We begin by describing the reconstruction procedure at time level t^n. In each cell, we define a linear polynomial as

$$\tilde{\rho}_\Delta^{k,n}(x, y) := ax + by + c, \quad \text{for} \quad (x, y) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}),$$

where a, b and c are constants. As the reconstructed polynomial $\tilde{\rho}_\Delta^{k,n}$ is expected to preserve the cell averages, we write

$$\tilde{\rho}_\Delta^{k,n}(x, y) = \rho_{ij}^{k,n} + a(x - x_i) + b(y - y_j), \tag{4.2.1}$$

where $a = \partial_x \tilde{\rho}^{k,n}_\Delta(x_i, y_j)$ and $b = \partial_y \tilde{\rho}^{k,n}_\Delta(x_i, y_j)$. Further, the slopes are determined using the minmod slope-limiter as

$$a = \sigma^{x,k,n}_{ij}/\Delta x \text{ and } b = \sigma^{y,k,n}_{ij}/\Delta y,$$

where

$$\begin{aligned}
\sigma^{x,k,n}_{ij} &:= 2\theta \text{minmod}\left((\rho^{k,n}_{i,j} - \rho^{k,n}_{i-1,j}),\ \frac{1}{2}(\rho^{k,n}_{i+1,j} - \rho^{k,n}_{i-1,j}),\ (\rho^{k,n}_{i+1,j} - \rho^{k,n}_{i,j}) \right), \\
\sigma^{y,k,n}_{ij} &:= 2\theta \text{minmod}\left((\rho^{k,n}_{i,j} - \rho^{k,n}_{i,j-1}),\ \frac{1}{2}(\rho^{k,n}_{i,j+1} - \rho^{k,n}_{i,j-1}),\ (\rho^{k,n}_{i,j+1} - \rho^{k,n}_{i,j}) \right),
\end{aligned} \tag{4.2.2}$$

for $\theta \in [0, 1]$. The minmod function is defined as

$$\text{minmod}(a_1, \cdots, a_m) := \begin{cases} \text{sgn}(a_1) \min\limits_{1 \le k \le m} \{|a_k|\} & \text{if } \text{sgn}(a_1) = \cdots = \text{sgn}(a_m), \\ 0 & \text{otherwise.} \end{cases}$$

The face values of the reconstructed polynomial in the x- and y-directions are given by

$$\rho^{k,n,-}_{i+\frac{1}{2},j} = \rho^{k,n}_{i,j} + \frac{\sigma^{x,k,n}_{i,j}}{2}, \quad \rho^{k,n,+}_{i-\frac{1}{2},j} = \rho^{k,n}_{i,j} - \frac{\sigma^{x,k,n}_{i,j}}{2} \tag{4.2.3}$$

and

$$\rho^{k,n,-}_{i,j+\frac{1}{2}} = \rho^{k,n}_{i,j} + \frac{\sigma^{y,k,n}_{i,j}}{2}, \quad \rho^{k,n,+}_{i,j-\frac{1}{2}} = \rho^{k,n}_{i,j} - \frac{\sigma^{y,k,n}_{i,j}}{2}, \tag{4.2.4}$$

respectively. Here, within each cell, the superscripts + and − indicate the left (bottom) and right (top) interfaces, respectively.

Given the cell-averaged solutions $\{\rho^{k,n}_{i,j}\}$, $k \in \{1, 2, \ldots, N\}$ at the time stage t^n, the fully discrete scheme involves two stages of the SSP Runge-Kutta method [103, 151] to compute the solution at the time level t^{n+1}. This is described as follows.

Step 1: Compute intermediate values $\rho^{k,(1)}_{ij}$ as

$$\begin{aligned}
\rho^{k,(1)}_{ij} = \rho^{k,n}_{ij} &- \lambda_x \left[F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i+\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i-\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) \right] \\
&- \lambda_y \left[G^{k,n}_{i,j+\frac{1}{2}}(\rho^{k,n,-}_{i,j+\frac{1}{2}} \rho^{k,n,+}_{i,j+\frac{1}{2}}) - G^{k,n}_{i,j-\frac{1}{2}}(\rho^{k,n,-}_{i,j-\frac{1}{2}}, \rho^{k,n,+}_{i,j-\frac{1}{2}}) \right],
\end{aligned} \tag{4.2.5}$$

where $F^{k,n}_{i\pm\frac{1}{2},j}$ and $G^{k,n}_{i,j\pm\frac{1}{2}}$ are numerical fluxes (defined as in Section 4.2.1). Next, reconstruct the piecewise linear polynomial from the values $\rho^{k,(1)}_{ij}$ as in (4.2.1) and compute the face values $\rho^{k,(1),\pm}_{i+\frac{1}{2},j}$ and $\rho^{k,(1),\pm}_{i,j+\frac{1}{2}}$ following (4.2.3) and (4.2.4).

Step 2: Compute intermediate values $\rho^{k,(2)}_{ij}$ as

$$\begin{aligned}
\rho^{k,(2)}_{ij} = \rho^{k,(1)}_{ij} &- \lambda_x \left[F^{k,(1)}_{i+\frac{1}{2},j}(\rho^{k,(1),-}_{i+\frac{1}{2},j}, \rho^{k,(1),+}_{i+\frac{1}{2},j}) - F^{k,(1)}_{i-\frac{1}{2},j}(\rho^{k,(1),-}_{i-\frac{1}{2},j}, \rho^{k,(1),+}_{i-\frac{1}{2},j}) \right] \\
&- \lambda_y \left[G^{k,(1)}_{i,j+\frac{1}{2}}(\rho^{k,(1),-}_{i,j+\frac{1}{2}} \rho^{k,(1),+}_{i,j+\frac{1}{2}}) - G^{k,(1)}_{i,j-\frac{1}{2}}(\rho^{k,(1),-}_{i,j-\frac{1}{2}}, \rho^{k,(1),+}_{i,j-\frac{1}{2}}) \right].
\end{aligned} \tag{4.2.6}$$

Finally, the solution at the $(n+1)$-th time-level is now computed as

$$\rho_{ij}^{k,n+1} = \frac{\rho_{ij}^{k,n} + \rho_{ij}^{k,(2)}}{2}. \tag{4.2.7}$$

We also write the approximate solution corresponding to the second-order scheme (4.2.7) as

$$\rho_{\Delta}^{k}(t,x,y) = \rho_{ij}^{k,n}, \quad \text{for} \quad (t,x,y) \in [t^n, t^{n+1}) \times [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}).$$

4.2.1 Numerical flux

The numerical fluxes in (4.2.5) are defined using a Lax-Friedrichs-type numerical flux considered in [14]:

$$\begin{aligned} F_{i+\frac{1}{2},j}^{k,n}(u,v) &:= \frac{f_{i+\frac{1}{2},j}^{k,n}(u) + f_{i+\frac{1}{2},j}^{k,n}(v)}{2} - \frac{\alpha(v-u)}{2\lambda_x}, \\ G_{i,j+\frac{1}{2}}^{k,n}(u,v) &:= \frac{g_{i,j+\frac{1}{2}}^{k,n}(u) + g_{i,j+\frac{1}{2}}^{k,n}(v)}{2} - \frac{\beta(v-u)}{2\lambda_y}, \end{aligned} \tag{4.2.8}$$

where $\alpha, \beta \in (0, \frac{1}{3(1+\theta)})$ are fixed constants and we define

$$f_{i+\frac{1}{2},j}^{k,n}(\rho) := f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho, \boldsymbol{A}_{i+\frac{1}{2},j}^n), \;\; g_{i,j+\frac{1}{2}}^{k,n}(\rho) := g^k(t^n, x_i, y_{j+\frac{1}{2}}, \rho, \boldsymbol{B}_{i,j+\frac{1}{2}}^n),$$

where the terms $\boldsymbol{A}_{i+\frac{1}{2},j}^n$ and $\boldsymbol{B}_{i,j+\frac{1}{2}}^n$ approximate the convolution terms as detailed in the following section.

4.2.2 Approximation of convolution terms

We define the approximated convolution terms:

$$\boldsymbol{A}_{i+\frac{1}{2},j}^n := \left(A_{i+\frac{1}{2},j}^{q,n}\right)_{q=1}^m \text{ and } \boldsymbol{B}_{i,j+\frac{1}{2}}^n := \left(B_{i,j+\frac{1}{2}}^{q,n}\right)_{q=1}^m,$$

where for each $q \in \{1, 2, \cdots, m\}$:

$$A_{i+\frac{1}{2},j}^{q,n} \approx (\boldsymbol{\rho} * \boldsymbol{\eta})_q \,(t^n, x_{i+\frac{1}{2}}, y_j), \quad B_{i,j+\frac{1}{2}}^{q,n} \approx (\boldsymbol{\rho} * \boldsymbol{\nu})_q (t^n, x_i, y_{j+\frac{1}{2}}).$$

Using the midpoint quadrature rule, we derive these approximations as as follows:

$$
\begin{aligned}
&(\boldsymbol{\rho} * \boldsymbol{\eta})_q\,(t^n, x_{i+\frac{1}{2}}, y_j) \\
&\quad = \sum_{k=1}^{N} \int\int_{\mathbb{R}^2} \eta^{q,k}(x_{i+\frac{1}{2}} - x', y_j - y')\rho^k(t, x', y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad = \sum_{k=1}^{N} \sum_{l,p\in\mathbb{Z}} \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \int_{y_{p-\frac{1}{2}}}^{y_{p+\frac{1}{2}}} \eta^{q,k}(x_{i+\frac{1}{2}} - x', y_j - y')\rho^k(t, x', y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad \approx \Delta x \Delta y \sum_{k=1}^{N} \left[\sum_{p,l\in\mathbb{Z}} \eta^{q,k}(x_{i+\frac{1}{2}} - x_l, y_j - y_p)\ \rho_{l,p}^{k,n} \right] \\
&\quad = \Delta x \Delta y \sum_{k=1}^{N} \left[\sum_{p,l\in\mathbb{Z}} \eta^{q,k}_{i+\frac{1}{2}-l,j-p}\ \rho_{l,p}^{k,n} \right] =: A^{q,n}_{i+\frac{1}{2},j},
\end{aligned}
\tag{4.2.9}
$$

and similarly:

$$
\begin{aligned}
&(\boldsymbol{\rho} * \boldsymbol{\nu})_q(t^n, x_i, y_{j+\frac{1}{2}}) \\
&\quad = \sum_{k=1}^{N} \int\int_{\mathbb{R}^2} \nu^{q,k}(x_i - x', y_{j+\frac{1}{2}} - y')\rho^k(t^n, x', y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad = \sum_{k=1}^{N} \sum_{l,p\in\mathbb{Z}} \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \int_{y_{p-\frac{1}{2}}}^{y_{p+\frac{1}{2}}} \nu^{q,k}(x_i - x', y_{j+\frac{1}{2}} - y')\rho^k(t^n, x', y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad \approx \Delta x \Delta y \sum_{k=1}^{N} \left[\sum_{l,p\in\mathbb{Z}} \nu^{q,k}_{i-l,j+\frac{1}{2}-p}\ \rho_{l,p}^{k,n} \right] =: B^{q,n}_{i,j+\frac{1}{2}},
\end{aligned}
$$

with the notations $\eta^{q,k}_{i+\frac{1}{2},j} := \eta^{q,k}(x_{i+\frac{1}{2}}, y_j)$ and $\nu^{q,k}_{i,j+\frac{1}{2}} := \nu^{q,k}(x_i, y_{j+\frac{1}{2}})$.

Remark 4.2.1. If we set $\theta = 0$ in the slope limiter (4.2.2), the scheme (4.2.7) reduces to a first-order in space and second-order in time scheme. In this case, a fully first-order scheme is obtained if we compute the approximate solution at t^{n+1} as

$$
\rho_{ij}^{k,n+1} = \rho_{ij}^{k,(1)}.
$$

4.3 Positivity-preserving property

We now show that the second-order scheme given by (4.2.7) admits a positivity-preserving property, i.e., for $n \in \mathbb{N} \cup \{0\}$, $\rho_{ij}^{k,n+1} \geq 0$ whenever $\rho_{ij}^{k,n} \geq 0$. To begin with, we write the Euler forward step (4.2.5) as the average

$$
\rho_{ij}^{k,(1)} = \frac{V_{ij}^{k,(1)} + W_{ij}^{k,(1)}}{2}, \tag{4.3.1}
$$

where

$$
V_{ij}^{k,(1)} := \rho_{ij}^{k,n} - 2\lambda_x \left[F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i+\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i-\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) \right] \tag{4.3.2}
$$

and

$$W_{ij}^{k,(1)} := \rho_{ij}^{k,n} - 2\lambda_y \left[G_{i,j+\frac{1}{2}}^{k,n}(\rho_{i,j+\frac{1}{2}}^{k,n,-}\rho_{i,j+\frac{1}{2}}^{k,n,+}) - G_{i,j-\frac{1}{2}}^{k,n}(\rho_{i,j-\frac{1}{2}}^{k,n,-}, \rho_{i,j-\frac{1}{2}}^{k,n,+}) \right].$$

Also, we note a useful property of the minmod reconstruction, given in the lemma below.

Lemma 4.3.1. *For a given k and n, if $\rho_{i,j}^{k,n} \geq 0 \ \forall \ i,j \in \mathbb{Z}$, then*

$$|\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+}| \leq 2\theta \rho_{ij}^{k,n}.$$

Proof. From the definition of slopes in (4.2.2), we obtain the inequalities:

$$0 \leq \frac{(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+})}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}}, \quad \frac{(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+})}{\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n}} \leq 2\theta.$$

Additionally, we observe that either $\rho_{i-1,j}^{k,n} < \rho_{i,j}^{k,n}$ or $\rho_{i+1j}^{k,n} < \rho_{i,j}^{k,n}$ provided $|\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+}| \neq 0$. Splitting into two cases and using the assumption $\rho_{i,j}^{k,n} \geq 0$, we obtain the following bounds:
Case 1: If $\rho_{i,j}^{k,n} > \rho_{i+1,j}^{k,n}$, then

$$|\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+}| = \frac{(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+})}{(\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n})} |\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n}| \leq 2\theta |\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n}| \leq 2\theta \rho_{ij}^{k,n}.$$

Case 2: If $\rho_{i,j}^{k,n} > \rho_{i-1,j}^{k,n}$, then

$$|\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+}| = \frac{(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,+})}{(\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n})} |\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}| \leq 2\theta |\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}| \leq 2\theta \rho_{ij}^{k,n}.$$

This completes the proof. □

Theorem 4.3.2. *Assume that the hypotheses **(H0)**, **(H1)** and **(H2)** hold and for all $k \in \{1, 2, \ldots, N\}$ the time-step Δt satisfies the following CFL conditions*

$$\bar{\lambda}_x \leq \frac{\min\{1, 4 - 6\bar{\alpha}(1+\theta), 6\bar{\alpha}\}}{\big(6(1+\theta)\|\partial_\rho f^k\| + 1\big)}, \quad \bar{\lambda}_y \leq \frac{\min\{1, 4 - 6\bar{\beta}(1+\theta), 6\bar{\beta}\}}{\big(6(1+\theta)\|\partial_\rho g^k\| + 1\big)}, \tag{4.3.3}$$

*where $\bar{\alpha} := 2\alpha$, $\bar{\beta} := 2\beta$, $\bar{\lambda}_x := 2\lambda_x$, $\bar{\lambda}_y := 2\lambda_y$ and the parameter $\theta \in [0,1]$ is as defined in the minmod slope-limiter (4.2.2). Additionally, assume that the mesh sizes are sufficiently small so that $\Delta x, \Delta y \leq \dfrac{1}{3M}$ where M is as in **(H1)**. If the initial datum $\boldsymbol{\rho_0} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2; \mathbb{R}_+^N)$, then the approximate solutions $\boldsymbol{\rho}_\Delta$ obtained using the second-order scheme (4.2.7) satisfy $\rho_\Delta^k(t,x,y) \geq 0$ for all $k \in \{1, 2, \ldots, N\}$, $t \in \mathbb{R}_+$ and $(x,y) \in \mathbb{R}^2$.*

Proof. To prove the positivity of the second-order scheme, we employ induction on the time index n. The base case for $n = 0$ holds trivially as the initial datum is non-negative, i.e., $\rho_{ij}^{k,0} \geq 0$ for all $i, j \in \mathbb{Z}$ and for all $k \in \{1, 2, \ldots, N\}$. For $n \geq 0$, it is required to show that $\rho_{ij}^{k,n+1} \geq 0$ whenever $\rho_{ij}^{k,n} \geq 0$. To do this, it suffices to prove that the forward Euler step (4.2.5) satisfies $\rho_{ij}^{k,(1)} \geq 0$ whenever $\rho_{ij}^{k,n} \geq 0$. This reduces to verifying that $V_{ij}^{k,(1)} \geq 0$, as the same argument applies to $W_{ij}^{k,(1)}$.

By adding and subtracting the term $\bar{\lambda}_x \left(F_{i+\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) - F_{i-\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) \right)$ in (4.3.2), $V_{ij}^{k,(1)}$ reads as

$$\begin{aligned} V_{ij}^{k,(1)} &= \rho_{i,j}^{k,n} - a_{i-\frac{1}{2},j}^{k,n}(\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}) + b_{i+\frac{1}{2},j}^{k,n}(\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n}) \\ &\quad - \bar{\lambda}_x \Big(F_{i+\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) - F_{i-\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) \Big) \\ &= \Big(1 - a_{i-\frac{1}{2},j}^{k,n} - b_{i+\frac{1}{2},j}^{k,n}\Big)\rho_{i,j}^{k,n} + a_{i-\frac{1}{2},j}^{k,n}\rho_{i-1,j}^{k,n} + b_{i+\frac{1}{2},j}^{k,n}\rho_{i+1,j}^{k,n} \\ &\quad - \bar{\lambda}_x \Big(F_{i+\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) - F_{i-\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) \Big), \end{aligned} \tag{4.3.4}$$

where

$$a_{i-\frac{1}{2},j}^{k,n} := \bar{\lambda}_x \tilde{a}_{i-\frac{1}{2},j}^{k,n} \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-}}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}} \right), \; b_{i+\frac{1}{2},j}^{k,n} := -\bar{\lambda}_x \tilde{b}_{i+\frac{1}{2},j}^{k,n} \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,+} - \rho_{i-\frac{1}{2},j}^{k,n,+}}{\rho_{i+1,j}^{k,n} - \rho_{i,j}^{k,n}} \right),$$

with

$$\begin{aligned} \tilde{a}_{i-\frac{1}{2},j}^{k,n} &:= \frac{\Big[F_{i-\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) - F_{i-\frac{1}{2},j}^{k,n}(\rho_{i-\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) \Big]}{(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-})}, \\ \tilde{b}_{i+\frac{1}{2},j}^{k,n} &:= \frac{\Big[F_{i+\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i+\frac{1}{2},j}^{k,n,+}) - F_{i+\frac{1}{2},j}^{k,n}(\rho_{i+\frac{1}{2},j}^{k,n,-}, \rho_{i-\frac{1}{2},j}^{k,n,+}) \Big]}{(\rho_{i+\frac{1}{2},j}^{k,n,+} - \rho_{i-\frac{1}{2},j}^{k,n,+})}. \end{aligned}$$

We will now show that $0 \leq a_{i-\frac{1}{2},j}^{k,n}, b_{i+\frac{1}{2},j}^{k,n} \leq \dfrac{1}{3}$. Observe that

$$0 \leq \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-}}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}} \right), \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,+} - \rho_{i-\frac{1}{2},j}^{k,n,+}}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}} \right) \leq (1+\theta),$$

where θ is as defined in the minmod limiter (4.2.2). From the definition of $F_{i+\frac{1}{2},j}^{k,n}$ in (4.2.8) and applying the mean value theorem, it follows that

$$\begin{aligned} a_{i-\frac{1}{2},j}^{k,n} &= \frac{\bar{\lambda}_x \hat{a}_{i-\frac{1}{2},j}^{k,n}}{2(\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-})} \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-}}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}} \right) \\ &= \left(\frac{\bar{\lambda}_x \partial_\rho f_{i-\frac{1}{2},j}^{k,n}(\zeta_{i-\frac{1}{2},j}^{k,n}) + \bar{\alpha}}{2} \right) \left(\frac{\rho_{i+\frac{1}{2},j}^{k,n,-} - \rho_{i-\frac{1}{2},j}^{k,n,-}}{\rho_{i,j}^{k,n} - \rho_{i-1,j}^{k,n}} \right) \\ &\leq \frac{\bar{\lambda}_x \|\partial_\rho f^k\| + \bar{\alpha}}{2}(1+\theta) \leq \frac{1}{3}, \end{aligned} \tag{4.3.5}$$

where $\zeta^{k,n}_{i-\frac{1}{2},j} \in \mathcal{I}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,-}_{i-\frac{1}{2},j})$, and we define

$$\hat{a}^{k,n}_{i-\frac{1}{2},j} := \left(f^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}) - f^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i-\frac{1}{2},j}) + \frac{\bar{\alpha}}{\bar{\lambda}_x}(\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,-}_{i-\frac{1}{2},j}) \right).$$

The final inequality in (4.3.5) follows from the bound

$$\bar{\lambda}_x\big(6(1+\theta)\|\partial_\rho f^k\| + 1\big) \leq 4 - 6\bar{\alpha}(1+\theta),$$

which is derived from the CFL condition (4.3.3). Furthermore, under the hypothesis **(H0)**, the inequality

$$\bar{\lambda}_x\big(6(1+\theta)\|\partial_\rho f^k\| + 1\big) \leq 6\bar{\alpha},$$

obtained from the CFL condition (4.3.3) yield

$$a^{k,n}_{i-\frac{1}{2},j} \geq \frac{-\bar{\lambda}_x\|\partial_\rho f^k_{i-\frac{1}{2},j}\| + \bar{\alpha}}{2} \left(\frac{\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,-}_{i-\frac{1}{2},j}}{\rho^{k,n}_{i,j} - \rho^{k,n}_{i-1,j}} \right) \geq 0.$$

In a similar way, we obtain the bound

$$0 \leq b^{k,n}_{i+\frac{1}{2},j} \leq \frac{1}{3}. \tag{4.3.6}$$

To estimate the last term of (4.3.4), we use the definition (4.2.8) and apply the triangle inequality, leading to

$$\left| F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) \right| \leq J_1 + J_2,$$

where we define

$$\begin{aligned} J_1 &:= \frac{1}{2}|f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j}) - f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j})|, \\ J_2 &:= \frac{1}{2}|f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j}) - f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j})|. \end{aligned} \tag{4.3.7}$$

Note that, by the choice of the slope limiter (4.2.2), the face values $\rho^{k,n,\pm}_{i\mp\frac{1}{2},j} \geq 0,\ \forall i,j \in \mathbb{Z}$. Further, we apply Lemma 4.3.1 to obtain

$$\begin{aligned} \rho^{k,n,-}_{i+\frac{1}{2},j} &= \rho^{k,n}_{i,j} + \frac{1}{2}(\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,+}_{i-\frac{1}{2},j}) \leq \rho^{k,n}_{i,j} + \frac{1}{2}|\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,+}_{i-\frac{1}{2},j}| \\ &\leq (1+\theta)\rho^{k,n}_{i,j}, \\ \rho^{n,+}_{i-\frac{1}{2},j} &= \rho^{k,n}_{i,j} - \frac{1}{2}(\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,+}_{i-\frac{1}{2},j}) \leq \rho^{k,n}_{i,j} + \frac{1}{2}|\rho^{k,n,-}_{i+\frac{1}{2},j} - \rho^{k,n,+}_{i-\frac{1}{2},j}| \\ &\leq (1+\theta)\rho^{k,n}_{i,j}. \end{aligned} \tag{4.3.8}$$

Now, by adding and subtracting $f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j})$ to the term J_1 of (4.3.7) and using the hypotheses **(H0)** and **(H1)** together with the expression (4.3.8), we obtain the following estimate:

$$
\begin{aligned}
J_1 &\le \frac{1}{2}\left(|f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j}) - f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j})|\right) \\
&\quad + \frac{1}{2}\left(|f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j})|\right) + \frac{1}{2}\left(|f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j})|\right) \\
&= \frac{1}{2}\left(|\partial_x f^k(t^n, \bar{x}_i, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j})|\Delta x\right) \\
&\quad + \frac{1}{2}\left(|\partial_\rho f^k(t^n, x_{i-\frac{1}{2}}, y_j, \bar{\rho}_i, \boldsymbol{A}^n_{i+\frac{1}{2},j})|\rho^{k,n,-}_{i+\frac{1}{2},j}\right) \\
&\quad + \frac{1}{2}\left(|\partial_\rho f^k(t^n, x_{i-\frac{1}{2}}, y_j, \hat{\rho}_i, \boldsymbol{A}^n_{i-\frac{1}{2},j})|\rho^{k,n,-}_{i+\frac{1}{2},j}\right) \\
&\le \left(\|\partial_\rho f^k\| + \frac{1}{2}M\Delta x\right)\rho^{k,n,-}_{i+\frac{1}{2},j} \le \left(\|\partial_\rho f^k\| + \frac{1}{2}M\Delta x\right)(1+\theta)\rho^{k,n}_{i,j}.
\end{aligned}
\tag{4.3.9}
$$

where $\bar{x}_i \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $\bar{\rho}_i, \hat{\rho}_i \in \mathcal{I}(0, \rho^{k,n,-}_{i+\frac{1}{2},j})$. The term J_2 is treated similarly to obtain

$$
J_2 \le (\|\partial_\rho f^k\| + \frac{1}{2}M\Delta x)\rho^{n,+}_{i-\frac{1}{2},j} \le (\|\partial_\rho f^k\| + \frac{1}{2}M\Delta x)(1+\theta)\rho^{k,n}_{i,j}. \tag{4.3.10}
$$

Combining the estimates (4.3.9) and (4.3.10), we get

$$
J_1 + J_2 \le \left(2(1+\theta)\|\partial_\rho f^k\| + M\Delta x\right)\rho^{k,n}_{i,j}. \tag{4.3.11}
$$

Next, in view of (4.3.11) we arrive at the estimate

$$
\begin{aligned}
&\bar{\lambda}_x\left|F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j})\right| \\
&\le \bar{\lambda}_x\left(2(1+\theta)\|\partial_\rho f^k\| + M\Delta x\right)\rho^{k,n}_{i,j} \le \frac{1}{3}\rho^{k,n}_{i,j},
\end{aligned}
\tag{4.3.12}
$$

where we use the conditions $\bar{\lambda}_x(6(1+\theta)\|\partial_\rho f^k\| + 1) \le 1$ (derived from (4.3.3)) and $\Delta x \le \dfrac{1}{3M}$. Thus, by invoking the estimates (4.3.5), (4.3.6) and (4.3.12) in (4.3.4), we obtain

$$
\begin{aligned}
V^{k,(1)}_{i,j} &\ge \left(1 - a^{k,n}_{i-\frac{1}{2},j} - b^{k,n}_{i+\frac{1}{2},j}\right)\rho^{k,n}_{i,j} + a^{k,n}_{i-\frac{1}{2},j}\rho^{k,n}_{i-1,j} + b^{k,n}_{i+\frac{1}{2},j}\rho^{k,n}_{i+1,j} \\
&\quad - \bar{\lambda}_x\left|F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j})\right| \\
&\ge \left(1 - a^{k,n}_{i-\frac{1}{2},j} - b^{k,n}_{i+\frac{1}{2},j} - \frac{1}{3}\right)\rho^{k,n}_{i,j} + a^{k,n}_{i-\frac{1}{2},j}\rho^{k,n}_{i-1,j} + b^{k,n}_{i+\frac{1}{2},j}\rho^{k,n}_{i+1,j} \ge 0.
\end{aligned}
$$

Following analogous arguments, we can show that $W^{k,(1)}_{ij} \ge 0$. Consequently, we deduce that $\rho^{k,(1)}_{ij} \ge 0$, for all $i, j \in \mathbb{Z}$. A similar treatment yields $\rho^{k,(2)}_{ij} \ge 0$, for all $i, j \in \mathbb{Z},$. Thus, from (4.2.7), we conclude that the final numerical solutions satisfy $\rho^{k,n+1}_{ij} \ge 0$, for $i, j \in \mathbb{Z}$, thereby completing the proof. □

Remark 4.3.3. It is immediate to check that the first-order scheme described in Remark 4.2.1 satisfies the positivity-preserving property under a CFL condition:

$$
\bar{\lambda}_x \le \frac{\min\{1, 4 - 6\bar{\alpha}, 6\bar{\alpha}\}}{(6\|\partial_\rho f^k\| + 1)}, \ \bar{\lambda}_y \le \frac{\min\{1, 4 - 6\bar{\beta}, 6\bar{\beta}\}}{(6\|\partial_\rho g^k\| + 1)}. \tag{4.3.13}
$$

Remark 4.3.4. The CFL conditions (4.3.13) and (4.3.3) corresponding to first- and second-order schemes, respectively impose constraints on the coefficients α and β that appear in the numerical flux (4.2.8). Specifically, the first-order scheme requires $\alpha, \beta \in \left(0, \frac{1}{3}\right)$, while the second-order scheme requires $\alpha, \beta \in \left(0, \frac{2}{9}\right)$.

We now present a corollary to Theorem 4.3.2, which will aid in proving the L^∞-stability. In the following, we denote $\boldsymbol{\rho}_\Delta(t) := \boldsymbol{\rho}_\Delta(t, \cdot, \cdot)$.

Corollary 4.3.5. *(L^1- stability) Under the assumptions of Theorem 4.3.2, for a non-negative initial datum $\boldsymbol{\rho}_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2; \mathbb{R}_+^N)$, the approximate solutions $\boldsymbol{\rho}_\Delta$ obtained from the scheme* (4.2.7) *satisfy*

$$\|\rho_\Delta^k(t)\|_{\mathrm{L}^1} = \|\rho_\Delta^k(0)\|_{\mathrm{L}^1}, \tag{4.3.14}$$

for all $k \in \{1, 2, \dots, N\}$ and $t \in \mathbb{R}_+$.

Proof. By Theorem (4.3.2), the non-negativity assumption on the initial datum imply that $\rho_{i,j}^{k,n} \geq 0$ for all $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Moreover, each stage in the Runge-Kutta time stepping we have $\rho_{i,j}^{k,(1)}, \rho_{i,j}^{k,(2)} \geq 0$ for all $i, j \in \mathbb{Z}$. Therefore, we obtain

$$\|\rho^{k,(1)}\|_{\mathrm{L}^1} = \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{k,(1)} = \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{k,n}$$

and

$$\|\rho^{k,(2)}\|_{\mathrm{L}^1} = \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{k,(2)} = \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{k,(1)}.$$

Consequently, we arrive at

$$\begin{aligned} \|\rho^{k,n+1}\|_{\mathrm{L}^1} = \Delta x \Delta y \sum_{i,j} \rho_{i,j}^{k,n+1} &= \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \frac{\rho_{i,j}^{k,(2)} + \rho_{i,j}^{k,n}}{2} \\ &= \Delta x \Delta y \sum_{i,j \in \mathbb{Z}} \rho_{i,j}^{k,n} = \|\rho^{k,n}\|_{\mathrm{L}^1}. \end{aligned}$$

The equality (4.3.14) now follows immediately. □

4.4 L^∞ stability

In this section, we establish that the second-order scheme given by (4.2.7) exhibits L^∞-stability.

Theorem 4.4.1. *(L^∞-stability) Let $\boldsymbol{\rho}_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2; \mathbb{R}_+^N)$. If the hypotheses **(H0)**, **(H1)** and **(H2)** and the CFL condition* (4.3.3) *hold along with the mesh-size restriction $\Delta x, \Delta y \leq \dfrac{1}{3M}$, then there exists a constant $C \geq 0$ depending only on $\boldsymbol{\rho}_0, \boldsymbol{\eta}, \boldsymbol{\nu}, \{f^k\}_{k=1}^N$ and $\{g^k\}_{k=1}^N$ such that the approximate solutions $\boldsymbol{\rho}_\Delta$ obtained from the second-order scheme (4.2.7) satisfy*

$$\|\boldsymbol{\rho}_\Delta(t)\| \leq \|\boldsymbol{\rho}_0\| e^{Ct},$$

for any $t \in \mathbb{R}_+$.

Proof. By Corollary 4.3.5 and applying the mean value theorem, we observe that the discrete convolutions (4.2.9) satisfy the following estimate:

$$\begin{aligned} \|\boldsymbol{A}^n_{i+\frac{1}{2},j} - \boldsymbol{A}^n_{i-\frac{1}{2},j}\| &= \left\| \left(\Delta x \Delta y \sum_{k=1}^{N} \sum_{p,l \in \mathbb{Z}} \left(\eta^{q,k}_{i+\frac{1}{2}-l,j-p} - \eta^{q,k}_{i-\frac{1}{2}-l,j-p} \right) \rho^{k,n}_{l,p} \right)_{q=1}^{m} \right\| \\ &\leq \Delta x \left(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(t^n)\|_{\mathrm{L}^1} \right) \\ &\leq \Delta x \left(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} \right). \end{aligned} \tag{4.4.1}$$

Further, invoking the estimate (4.4.1) and using the hypotheses **(H0)** and **(H1)**, we obtain

$$\begin{aligned} &\left| F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) \right| \\ &\leq \frac{1}{2} \left| \left(f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j}) - f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j}) \right) \right| \\ &\quad + \frac{1}{2} \left| \left(f^k(t^n, x_{i+\frac{1}{2}}, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \boldsymbol{A}^n_{i+\frac{1}{2},j}) - f^k(t^n, x_{i-\frac{1}{2}}, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j}) \right) \right| \\ &\leq \frac{1}{2} |\partial_x f^k(t^n, \bar{x}_i, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \bar{\boldsymbol{A}}^n_{i,j}) \Delta x| \\ &\quad + \frac{1}{2} \left(\|\nabla_A f^k(t^n, \bar{x}_i, y_j, \rho^{k,n,-}_{i+\frac{1}{2},j}, \bar{\boldsymbol{A}}^n_{i,j})\| \|\boldsymbol{A}^n_{i+\frac{1}{2},j} - \boldsymbol{A}^n_{i-\frac{1}{2},j}\| \right) \\ &\quad + \frac{1}{2} |\partial_x f^k(t^n, \tilde{x}_i, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \tilde{\boldsymbol{A}}^n_{i,j}) \Delta x| \\ &\quad + \frac{1}{2} \left(\|\nabla_A f^k(t^n, \tilde{x}_i, y_j, \rho^{k,n,+}_{i-\frac{1}{2},j}, \tilde{\boldsymbol{A}}^n_{i,j})\| \|\boldsymbol{A}^n_{i+\frac{1}{2},j} - \boldsymbol{A}^n_{i-\frac{1}{2},j}\| \right) \\ &\leq \frac{1}{2} \left(M \rho^{k,n,-}_{i+\frac{1}{2},j} \Delta x \Big(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1 \Big) \right) \\ &\quad + \frac{1}{2} \left(M \rho^{k,n,+}_{i-\frac{1}{2},j} \Delta x \Big(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1 \Big) \right) \\ &= M \rho^{k,n}_{i,j} \Delta x \left(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1 \right), \end{aligned} \tag{4.4.2}$$

where $\bar{x}_i, \tilde{x}_i \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $\bar{\boldsymbol{A}}^n_{i,j}, \tilde{\boldsymbol{A}}^n_{i,j} \in \tilde{\mathcal{I}}(\boldsymbol{A}^n_{i+\frac{1}{2},j}, \boldsymbol{A}^n_{i-\frac{1}{2},j})$. Now, in view of the estimates (4.3.5), (4.3.6) and (4.4.2), the terms $V^{k,(1)}_{ij}$ in (4.3.4) can be bounded as

$$\begin{aligned} |V^{k,(1)}_{ij}| &\leq \Big(1 - a^{k,n}_{i-\frac{1}{2},j} - b^{k,n}_{i+\frac{1}{2},j} \Big) |\rho^{k,n}_{i,j}| + a^{k,n}_{i-\frac{1}{2},j} |\rho^{k,n}_{i-1,j}| + b^{k,n}_{i+\frac{1}{2},j} |\rho^{k,n}_{i+1,j}| \\ &\quad + \bar{\lambda}_x \left| F^{k,n}_{i+\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) - F^{k,n}_{i-\frac{1}{2},j}(\rho^{k,n,-}_{i+\frac{1}{2},j}, \rho^{k,n,+}_{i-\frac{1}{2},j}) \right| \\ &\leq \|\rho^k_\Delta(t^n)\| \Big(1 + 2M\Delta t \Big(\|\partial_x \boldsymbol{\eta}\| \|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1 \Big) \Big). \end{aligned} \tag{4.4.3}$$

An analogous argument for $W_{ij}^{k,(1)}$ yields

$$|W_{ij}^{k,(1)}| \leq \|\rho_\Delta^k(t^n)\|\Big(1 + 2M\Delta t\Big(\|\partial_y \boldsymbol{\nu}\|\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big). \tag{4.4.4}$$

Consequently, using the bounds (4.4.3) and (4.4.4) in (4.3.1), it follows that

$$|\rho_{ij}^{k,(1)}| \leq \|\rho_\Delta^k(t^n)\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big).$$

Similar arguments for the second forward Euler step (4.2.6) give us the estimate

$$\begin{aligned}|\rho_{ij}^{k,(2)}| &\leq \|\rho_\Delta^{k,(1)}\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big) \\ &\leq \|\rho_\Delta^k(t^n)\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big)^2.\end{aligned} \tag{4.4.5}$$

Finally, in light of the estimate (4.4.5), we deduce that

$$\begin{aligned}|\rho_{ij}^{k,n+1}| &= \frac{1}{2}(|\rho_{ij}^{k,n}| + |\rho_{ij}^{k,(2)}|) \\ &\leq \|\rho_\Delta^k(t^n)\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big)^2 \\ &\leq \|\rho_\Delta^k(t^{n-1})\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big)^4 \\ &\ \ \vdots \\ &\leq \|\rho_\Delta^k(0)\|\Big(1 + 2M\Delta t\Big(\max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_\Delta(0)\|_{\mathrm{L}^1} + 1\Big)\Big)^{2(n+1)} \\ &\leq \|\boldsymbol{\rho}_\Delta(0)\|e^{Ct},\end{aligned} \tag{4.4.6}$$

for $t = (n+1)\Delta t$, where $C := 4M\Big(1 + \max\{\|\partial_x \boldsymbol{\eta}\|, \|\partial_y \boldsymbol{\nu}\|\}\|\boldsymbol{\rho}_0\|_{\mathrm{L}^1}\Big)$. The estimate (4.4.6) completes the proof. □

4.5 Numerical experiments

In this section, we present the results of our numerical experiments for the proposed second-order scheme and compare them with the first-order scheme described in Remark 4.2.1. Additionally, we demonstrate that the numerical results corresponding to the second-order scheme validate the positivity-preserving property established theoretically. We consider two types of problems in two space dimensions: a crowd dynamics model governed by a scalar non-local equation and the Keyfits-Kranzer system, both adhering to the framework of (4.1.1). In all the numerical results, the time-step Δt is determined from the CFL condition (4.3.3) associated with the second-order scheme. The computational domain $[x_1, x_2] \times [y_1, y_2]$ is discretized into $(n_x \times n_y)$ number of Cartesian cells, with grid sizes defined as $\Delta x := (x_2 - x_1)/n_x$ and $\Delta y := (y_2 - y_1)/n_y$. We set $\theta = 0.5$ in computing the slopes (4.2.2), and choose $\alpha = \beta = 1/6$ in the numerical fluxes (4.2.8). The initial and boundary conditions are specified in the description of each example. Hereafter, we refer to the first-order scheme described in Remark 4.2.1 as FO, and the second-order scheme (4.2.7) as SO.

Example 4.1. In this example, we consider the two-dimensional macroscopic crowd dynamics problem studied in [14], where the density of pedestrians ρ is modeled to evolve according to the scalar non-local conservation law:

$$\partial_t \rho + \nabla \cdot (\rho(1-\rho)(1-\rho * \mu)\vec{\boldsymbol{v}}) = 0, \tag{4.5.1}$$

with the convolution defined as

$$\rho * \mu(t,x,y) = \int\int_{\mathbb{R}^2} \mu(x-x', y-y')\rho(t,x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

The smooth kernel function μ quantifies the weight assigned by pedestrians to their surrounding crowd density, while the vector field $\vec{\boldsymbol{v}}(x,y) = (v^1(x,y), v^2(x,y))^T$ describes the path they follow. It is evident that (4.5.1) aligns with the framework of (4.1.1) (see Lemma 3.1 in [14]). We examine a scenario where two groups of individuals start from two different locations within the domain $[0,10] \times [-1,1]$, move in the same direction and eventually stop at the spot $\{9.5\} \times [-1,1]$. To account for this dynamics, the velocity vector field is chosen as

$$\vec{\boldsymbol{v}}(x,y) = \begin{bmatrix} (1-y^2)^3 \mathrm{exp}(-1/(x-9.5)^2)\chi_{(-\infty,9.5]\times[-1,1]}(x,y) \\ -2y\mathrm{exp}(1-1/y^2) \end{bmatrix},$$

where $\Omega \subseteq \mathbb{R}^2$ and χ_Ω denotes the indicator function of Ω. Further, the kernel function is defined to be of compact support in a disk of radius $r = 0.4$, centered at the origin:

$$\mu(x,y) = \frac{\tilde{\mu}(x,y)}{\int_{\mathbb{R}^2} \tilde{\mu}(x,y)\,\mathrm{d}x\,\mathrm{d}y}, \tag{4.5.2}$$

where

$$\tilde{\mu}(x,y) = (0.16 - x^2 - y^2)^3 \chi_{\{(x,y): x^2+y^2 \le 0.16\}}(x,y).$$

Note that the kernel function μ in (4.5.2) attains a global maximum at the origin $(0,0)$ and decreases radially, reflecting the fact that pedestrians prioritize nearby crowd density over distant ones. We solve the problem (4.5.1) with the initial datum:

$$\rho^0(x,y) = \chi_{[1,4]\times[0.1,0.8]}(x,y) + \chi_{[2,5]\times[-0.8,-0.1]}(x,y) \tag{4.5.3}$$

as shown in Fig.4.1, and impose 'no flow' boundary conditions (implemented by setting the numerical flux to zero at the boundary interfaces) on all sides of the computational domain $[0,10] \times [-1,1]$. Throughout this example, we use a common time-step for both the FO and SO schemes, setting $\Delta t = 0.026\Delta x$. This choice of time-step is derived from

(4.3.3), utilizing the bound $\|\partial_\rho f\|, \|\partial_\rho g\| \leq 2$ specific to this example. We first compute the solution at time $t = 4.0$ using both the FO and SO schemes, and demonstrate that the FO scheme solutions converge toward the SO scheme solutions as the mesh is refined. This comparison is illustrated in Fig. 4.2, where subfigures (A), (B), and (C) display the FO scheme solutions at progressively finer meshes, while subfigure (D) presents the SO scheme solution. The results clearly indicate that the FO scheme requires at least a fourfold refinement in mesh size to achieve a solution profile comparable to that of the SO scheme, thereby emphasizing the significance of the SO scheme.

In Fig. 4.3, we display the numerical solutions at various time levels, $t \in \{8.0, 12.0, 16.0, 20.0\}$, computed using both the FO and SO schemes with the same initial datum as in (4.5.3). A comparison of the solution profiles reveals significant differences between the results obtained from the FO and SO schemes. Furthermore, in Fig. 4.4, we provide the 1-D plots of the SO scheme solutions given in Fig. 4.3 (B), (D), (F) and (H), along the diagonals $y = \frac{x}{5} - 1$ and $y = 1 - \frac{x}{5}$. These plots indicate that the solutions generated using the SO scheme remain positive and exhibit L^∞-stability, thereby confirming the theoretical results.

Example 4.2. We compute the experimental order of convergence (E.O.C.) for both the FO and SO schemes using the problem (4.5.1) and initial condition (4.5.3) described in Example 4.1, and compare their performance. For uniform grids, we denote $h := \Delta x = \Delta y$. Since the exact solution to the problem (4.5.1) with the initial condition (4.5.3) is not known, the E.O.C. is estimated based on the L^1-error between numerical solutions computed on successive mesh refinements with grid sizes h, $h/2$, and so on. The formula for computing E.O.C. is given by

$$\gamma := \log\left(\frac{\|\rho_h - \rho_{\frac{h}{2}}\|_{\mathrm{L}^1}}{\|\rho_{\frac{h}{2}} - \rho_{\frac{h}{4}}\|_{\mathrm{L}^1}}\right) / \log 2,$$

where, ρ_h denotes the numerical solution on the mesh of size h. The numerical solutions are computed up to time $t = 0.2$ for mesh size $h \in \{0.05, 0.025, 0.00125, 0.00625, 0.003125\}$ in the computational domain $[0, 10] \times [-1, 1]$. Both the FO and SO schemes solutions are computed with the same time step $\Delta t = 0.026\Delta x$. The results, summarized in Table 4.1, indicate that the FO scheme achieves an E.O.C. of approximately $\gamma \approx 0.5$, while the SO scheme attains an E.O.C. of approximately $\gamma \approx 0.8$. The relatively low E.O.C. observed for both the FO and SO schemes is attributed to the possible non-smooth nature of the exact solution. Nevertheless, the SO scheme demonstrates nearly twofold better accuracy.

Example 4.3. We consider the non-local Keyfitz-Kranzer (KK) system introduced in [14], which extends the classical Keyfitz-Kranzer system from [122] to a non-local framework. This two-dimensional system involves two unknowns, i.e., $N = 2$, with $\boldsymbol{\rho} = (\rho^1, \rho^2)$, and is

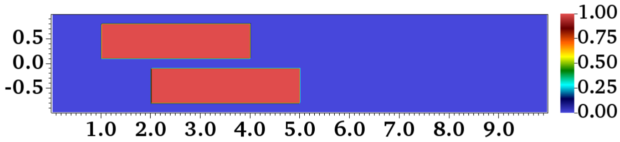
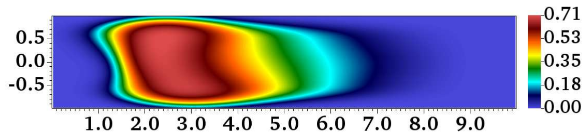
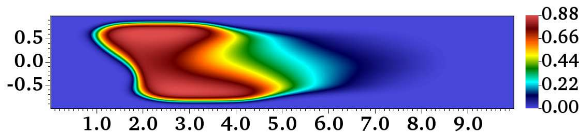


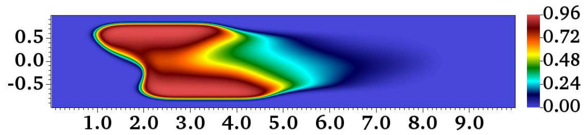
Figure 4.1: Example 4.1: initial condition ρ^0 for the problem (4.5.1).



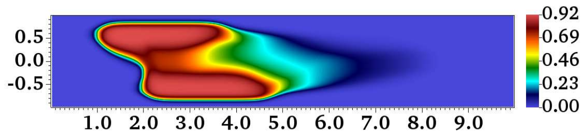
(a) FO, $\Delta x = \Delta y = 0.025 (400 \times 80)$



(b) FO, $\Delta x = \Delta y = 0.0125\,(800 \times 160)$



(c) FO, $\Delta x = \Delta y = 0.00625 \;\; (1600 \times 320)$



(d) SO, $\Delta x = \Delta y = 0.025\,(400 \times 80)$

Figure 4.2: Example 4.1: numerical solutions of the problem (4.5.1), computed with initial condition (4.1). The results in (A), (B) and (C) are obtained using the FO scheme with meshes of resolution (400×80), (800×160) and (1600×320), respectively. The SO scheme solution is displayed in (D), computed on a mesh of resolution (400×80). All results are shown at time $t = 4.0$ with a common time step $\Delta t = 0.026\Delta x$.

Table 4.1: Example 4.2: numerical errors produced by the FO and SO schemes, applied to the problem (4.5.1) with initial datum (4.5.3). The results are obtained at $t = 0.2$ with a common time step of $\Delta t = 0.026\Delta x$.

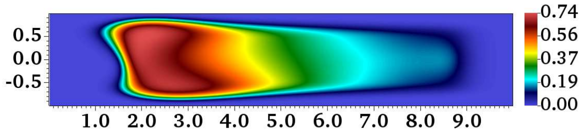
	FO scheme		SO scheme	
h	$\lVert\rho_h - \rho_{\frac{h}{2}}\rVert_{\mathrm{L}^1}$	γ	$\lVert\rho_h - \rho_{\frac{h}{2}}\rVert_{\mathrm{L}^1}$	γ
0.05	0.63622	0.3036201	0.506055	0.6217728
0.025	0.5154761	0.3999979	0.3288709	0.7782156
0.0125	0.3906584	0.4629401	0.1917605	0.7862285
0.00625	0.2834251	-	0.1111939	-

given by

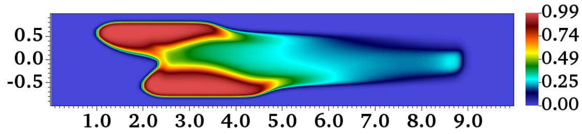
$$\begin{aligned} \partial_t \rho^1 + \partial_x(\rho^1 \varphi^1(\mu * \rho^1, \mu * \rho^2)) + \partial_y(\rho^1 \varphi^2(\mu * \rho^1, \mu * \rho^2)) = 0, \\ \partial_t \rho^2 + \partial_x(\rho^2 \varphi^1(\mu * \rho^1, \mu * \rho^2)) + \partial_y(\rho^2 \varphi^2(\mu * \rho^1, \mu * \rho^2)) = 0, \end{aligned} \tag{4.5.4}$$

where the functions φ^1 and φ^2 are defined as

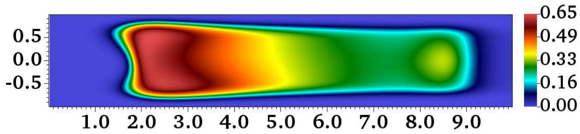
$$\varphi^1(A_1, A_2) := \sin(A_1^2 + A_2^2), \quad \varphi^2(B_1, B_2) := \cos(B_1^2 + B_2^2),$$



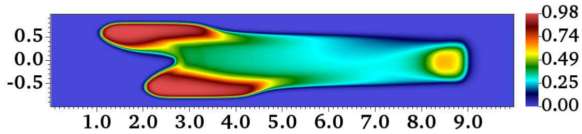
(a) ρ at time $t = 8$, FO-scheme, $\Delta x = \Delta y = 0.0125$



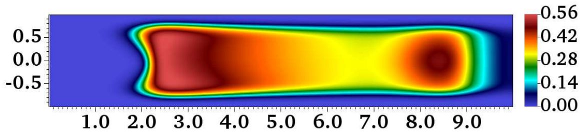
(b) ρ at time $t = 8$, SO-scheme, $\Delta x = \Delta y = 0.0125$



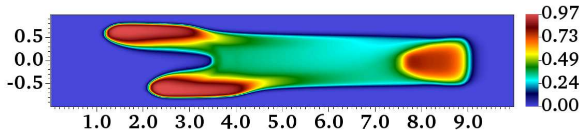
(c) ρ at time $t = 12$, FO-scheme, $\Delta x = \Delta y = 0.0125$



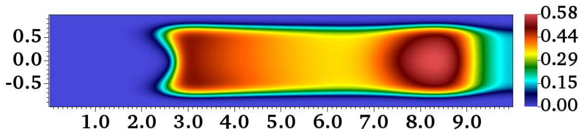
(d) ρ at time $t = 12$, SO-scheme, $\Delta x = \Delta y = 0.0125$



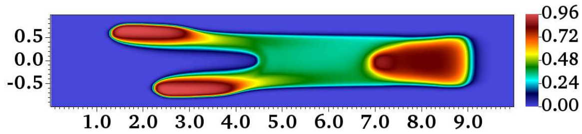
(e) ρ at time $t = 16$, FO-scheme, $\Delta x = \Delta y = 0.0125$



(f) ρ at time $t = 16$, SO-scheme, $\Delta x = \Delta y = 0.0125$



(g) ρ at time $t = 20$, FO-scheme, $\Delta x = \Delta y = 0.0125$



(h) ρ at time $t = 20$, SO-scheme, $\Delta x = \Delta y = 0.0125$

Figure 4.3: Example 4.1: approximate solutions of the problem (4.5.1) with initial datum (4.5.3), computed using the FO scheme (A, C, E, G) and the SO scheme (B, D, F, H). All results are obtained using a common time step $\Delta t = 0.026\Delta x$.

and the kernel function μ is given by

$$\mu(x, y) = \frac{\tilde{\mu}(x, y)}{\int_{\mathbb{R}^2} \tilde{\mu}(x, y)\,\mathrm{d}x\,\mathrm{d}y}, \qquad \text{where } \tilde{\mu} = \left(r^2 - (x^2 + y^2)\right)^3 \chi_{\{(x,y):x^2+y^2 \leq r^2\}}(x, y)$$

and $r > 0$ denotes the radius of the support of μ. The system (4.5.4) fits into the general framework of system (4.1.1), with flux functions in the form

$$\begin{aligned} f^k(t, x, y, \rho^k, \boldsymbol{\eta} * \boldsymbol{\rho}) &:= \rho^k \varphi^1(\mu * \rho^1, \mu * \rho^2), \\ g^k(t, x, y, \rho^k, \boldsymbol{\nu} * \boldsymbol{\rho}) &:= \rho^k \varphi^2(\mu * \rho^1, \mu * \rho^2), \quad k \in \{1, 2\}, \end{aligned}$$

where the kernel matrices $\boldsymbol{\eta}$ and $\boldsymbol{\nu}$ are given by

$$\boldsymbol{\eta} = \boldsymbol{\nu} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

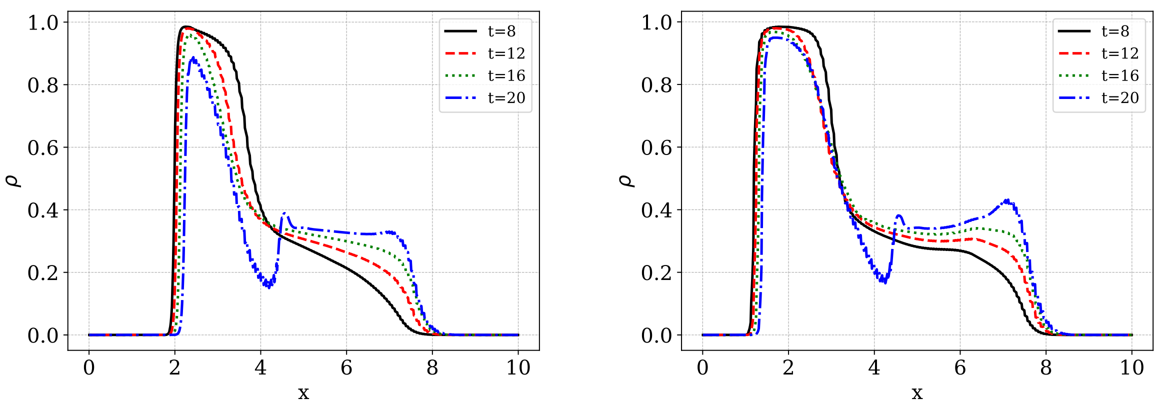


Figure 4.4: Example 4.1: 1-D plot of the SO solutions from Figures 4.3 (B), (D), (F) and (H). The solution along the diagonals $y = \frac{x}{5} - 1$ is shown in (A) and the solution along the diagonal $y = 1 - \frac{x}{5}$ is given in (B).

and the convolution terms simplify to

$$\boldsymbol{\eta} * \boldsymbol{\rho} = \boldsymbol{\nu} * \boldsymbol{\rho} = \left(\mu * \rho^1, \mu * \rho^2\right).$$

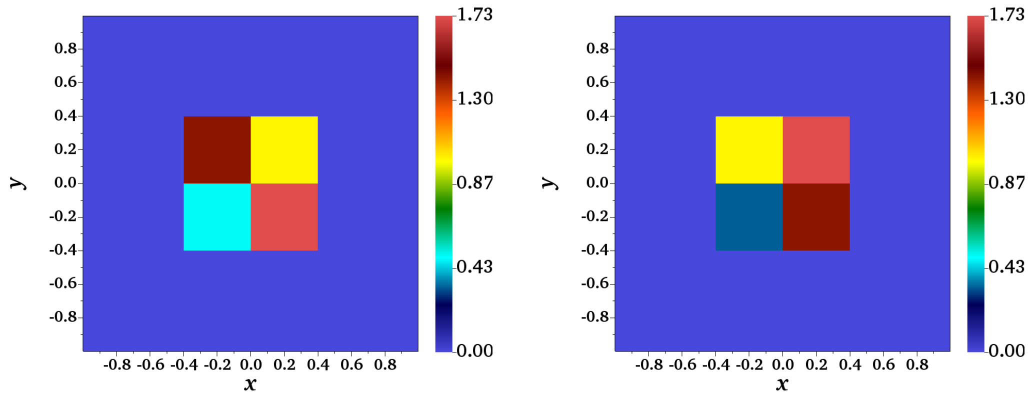
We conduct numerical simulations of the problem (4.5.5) using the initial condition given by

$$\boldsymbol{\rho}_0(x,y) = \left(\rho_0^1(x,y), \rho_0^2(x,y)\right) = \begin{cases} (1, \sqrt{3}), & (x,y) \in (0, 0.4] \times (0, 0.4], \\ (\sqrt{2}, 1), & (x,y) \in [-0.4, 0] \times (0, 0.4], \\ \left(\frac{1}{2}, \frac{1}{3}\right), & (x,y) \in [-0.4, 0] \times [-0.4, 0], \\ (\sqrt{3}, \sqrt{2}), & (x,y) \in (0, 0.4] \times [-0.4, 0], \\ (0,0), & \text{elsewhere}, \end{cases} \tag{4.5.5}$$

as illustrated in Fig. 4.5. The computational domain is taken as $[-1, 1] \times [-1, 1]$, with out flow boundary conditions applied along all sides of the domain. The kernel radius is set to $r = 0.0125$, and the approximate solutions (ρ_1, ρ_2) are evolved up to time $t = 0.1$. We compare the solutions obtained using the FO and SO schemes for different spatial resolutions, with time step $\Delta t = 0.05\Delta x$. This time-step is chosen based on the CFL condition (4.3.3), noting that

$$\|\partial_\rho f^k\|, \|\partial_\rho g^k\| \leq 1, \text{ for } k = 1, 2,$$

in this example. In Fig. 4.6, the FO scheme results are computed on a (1600×1600) mesh, while the SO scheme solutions are computed on a coarser (800×800) mesh. Similarly, in Fig. 4.7, we compare the FO scheme solutions on a (3200×3200) mesh with the SO scheme solutions on a (1600×1600) mesh. These results clearly illustrate that the SO



(a) ρ^1 (b) ρ^2

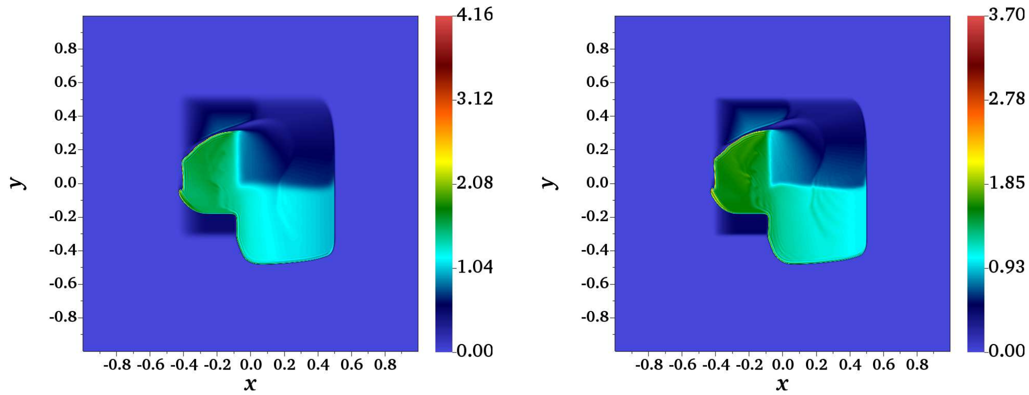
Figure 4.5: Example 4.3: initial condition (4.5.5) for the KK system (4.5.4).

scheme produces comparable solutions with half of the mesh size required by the FO scheme. This highlights the effectiveness of the SO scheme solving the given problem numerically. Furthermore, numerical results confirm that the SO scheme preserves the positivity property and satisfies L^∞-stability, in agreement with the theoretical analysis.

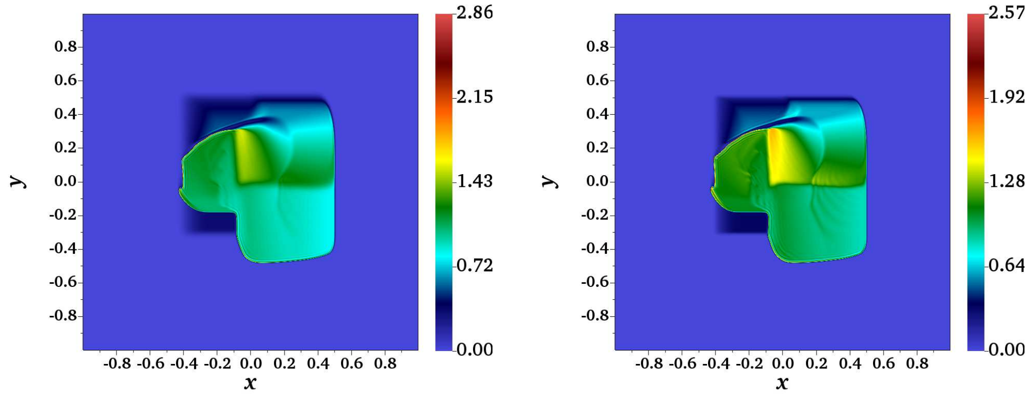
Example 4.4. In this example, we consider the non-local Keyfitz-Kranzer model (4.5.4) and study the behavior of solutions as the radius of the convolution kernels approaches zero, which is equivalent to the convolution kernels converging to the Dirac delta distribution. This problem, known as the 'singular limit problem' has been explored numerically in [19, 14], and theoretical results for specific cases have been established in [60, 64, 65]. However, analytical convergence results for the general case remain an open problem. It is desirable that numerical schemes that approximate non-local models retain their robustness under variations in model parameters. A recent study in this direction is available in [110]. In view of this, we examine the behavior of both the FO and SO schemes in the singular limit regime. The corresponding local version of the Keyfitz-Kranzer system is given by

$$\begin{aligned}
\partial_t \rho^1 + \partial_x(\rho^1 \varphi^1(\rho^1,\rho^2)) + \partial_y(\rho^1 \varphi^2(\rho^1,\rho^2)) &= 0,\\
\partial_t \rho^2 + \partial_x(\rho^2 \varphi^1(\rho^1,\rho^2)) + \partial_y(\rho^2 \varphi^2(\rho^1,\rho^2)) &= 0.
\end{aligned} \tag{4.5.6}$$

We perform this analysis for convolution kernel radii $r \in \{0.04, 0.02, 0.01, 0.005, 0.0025\}$ at different time levels $t \in \{0.03, 0.07, 0.1\}$. We compute the L^1 distance between the solutions corresponding to the non-local (4.5.4) and local (4.5.6) versions of the KK system, using the same initial condition as in (4.5.5). The non-local solutions are computed on a (1600×1600) mesh, while the local model (4.5.6) is solved on a finer (3200×3200) mesh, with both computations performed using the SO scheme. In all the computations, the time step is fixed as $\Delta t = 0.05\Delta x$, and the boundary conditions are same as those used in Example 4.3. The results displayed in Table 4.2 indicate that the SO scheme solutions converge to the local version as the parameter r approaches zero. Furthermore, we observe



(a) ρ^1(FO-scheme, (1600×1600) cells) (b) ρ^1(SO-scheme, (800×800) cells)



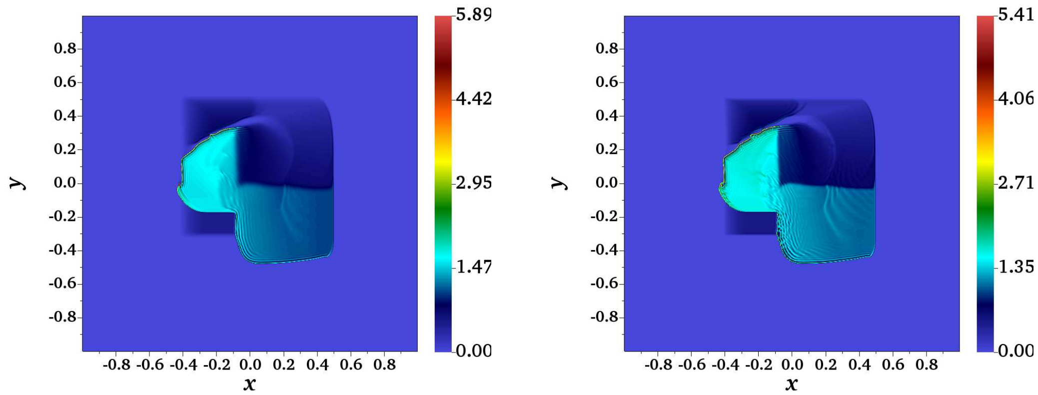
(c) ρ^2(FO-scheme, (1600×1600) cells) (d) ρ^2(SO-scheme, (800×800) cells)

Figure 4.6: Example 4.3: numerical solutions ρ^1 and ρ^2 of the KK system (4.5.4) with the initial condition (4.5.5), computed at time $t = 0.1$ using (A, C) the FO scheme and (B, D) the SO scheme. The FO scheme solutions are computed with a mesh resolution (1600×1600), while SO scheme uses a resolution of (800×800). All results are computed using time step $\Delta t = 0.05\Delta x$, and the kernel function parameter is set to $r = 0.0125$.

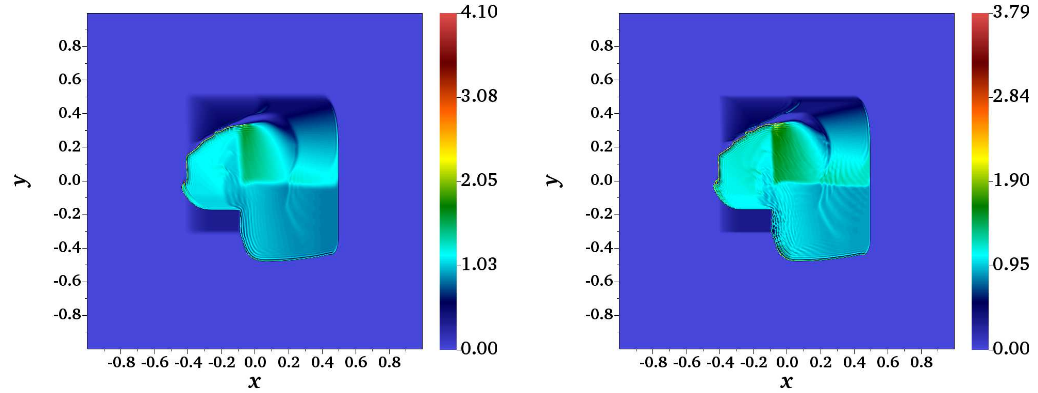
that the rate of convergence of the SO scheme is higher than that of the FO scheme.

4.6 Concluding remarks

In this work, we propose a fully discrete second-order scheme for a general system of non-local conservation laws in multiple space dimensions. The scheme is theoretically proven to be positivity-preserving and L^∞-stable. Numerical experiments clearly demonstrate the superior accuracy of the SO scheme compared to its first-order counterpart, as illustrated in Figs. 4.2 and 4.3 for a non-local scalar case, and Fig. 4.6 and 4.7 for a system case. Furthermore, the numerical results in all figures confirm the theoretical properties of positivity preservation and L^∞ stability. To illustrate this behavior more clearly, we include in Fig. 4.4 the one-dimensional cross sections of the SO scheme solutions displayed in Fig. 4.3 (B), (D), (F) and (H). We have also computed the experimental order of



(a) ρ^1(FO- scheme,(3200×3200) cells) (b) ρ^1(SO-scheme, (1600×1600) cells)



(c) ρ^2(FO-scheme,(3200×3200) cells) (d) ρ^2(SO- scheme,(1600×1600) cells)

Figure 4.7: Example 4.3: numerical solutions ρ^1 and ρ^2 of the KK system (4.5.4) with the initial condition (4.5.5), computed at time $t = 0.1$ using (A, C) the FO scheme and (B, D) the SO scheme. The time step is set as $\Delta t = 0.05\Delta x$, and the parameter of the kernel function is taken as $r = 0.0125$. The FO scheme solutions are computed on mesh of resolution (3200×3200), while SO scheme uses a resolution of (1600×1600).

convergence of the SO scheme in the scalar case and compared it with the FO scheme, as shown in Table 4.1. We do not observe an E.O.C. close to two for the SO scheme, as the underlying solution is not expected to be smooth. Nevertheless, we obtain an E.O.C. that is nearly twice that of the FO scheme, confirming its enhanced accuracy. The robustness of the SO scheme is further evaluated in the context of the 'singular limit problem' and the results show that the SO scheme solutions approach the local problem as the parameter r tends to zero, with a higher convergence rate compared to that of FO scheme, as is evident from Table 4.2.

Table 4.2: Example 4.4: L^1 distance between the solutions of the non-local (4.5.4) and local (4.5.6) versions of the KK system with initial condition (4.5.5), computed using both the FO and SO schemes on a mesh of resolution (1600×1600). The solutions to the local problem are computed on a mesh of (3200×3200) cells using the SO scheme. The kernel radii are chosen as $r \in \{0.04, 0.02, 0.01, 0.005, 0.0025\}$, and solutions are computed at times $t \in \{0.03, 0.07, 0.1\}$ with a time step $\Delta t = 0.05\Delta x$.

Scheme		ρ^1			ρ^2		
	r \ t	0.03	0.07	0.1	0.03	0.07	0.1
FO	0.04	0.0937	0.1446	0.1575	0.0837	0.1344	0.1376
	0.02	0.0576	0.0836	0.0882	0.0519	0.0790	0.0808
	0.01	0.0384	0.0519	0.0531	0.0344	0.0493	0.0496
	0.005	0.0250	0.0317	0.0317	0.0225	0.0297	0.0293
	0.0025	0.0179	0.0226	0.0239	0.0169	0.0208	0.0216
SO	0.04	0.1323	0.2379	0.2839	0.1223	0.2262	0.2586
	0.02	0.0843	0.1373	0.1511	0.0789	0.1327	0.1392
	0.01	0.0492	0.0749	0.0807	0.0462	0.0750	0.0787
	0.005	0.0288	0.0390	0.0410	0.0263	0.0389	0.0395
	0.0025	0.0115	0.0138	0.0137	0.0100	0.0133	0.0129

5

A MUSCL-type central scheme for conservation laws with discontinuous flux

We are interested in the second-order discretization of the Cauchy problem for scalar conservation laws with spatially varying flux:

$$\begin{aligned} u_t + f(k(x), u)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{5.0.1}$$

where t and x are the time and space variables, respectively and $u = u(x,t)$ is the unknown quantity. Here, the coefficient $k(x)$ in the flux function f is allowed to be a discontinuous function of the spatial variable x.

We refer to Chapter 1, Section 1.2, for a brief review of numerical techniques for (5.0.1). As discussed there, the analysis of second-order numerical methods is particularly challenging; the only existing attempts at provably convergent second-order schemes are [44, 4]. Nonetheless, these studies rely on a non-local limiter algorithm to ensure the scheme is FTVD (flux total variation diminishing). While effective, the limiter algorithm introduces additional computational tasks compared to conventional second-order schemes. This naturally leads to a question: Is it possible to design a relatively simple scheme, such as one based on MUSCL-type spatial reconstruction, and establish its convergence to the entropy solution? To the best of our knowledge, this remains an open question, as also noted in [44]. In this work, we aim to address this problem by proposing and analyzing a comparatively simple second-order scheme, specifically, a variant of the Nessyahu-Tadmor central scheme (see [137]) that employs the minmod limiter for reconstruction.

While upwind schemes provide higher-resolution solutions, they are often more restrictive due to the need for solving Riemann problems, either exactly or approximately, at mesh interfaces. In contrast, central schemes, such as the Lax-Friedrichs (LF) scheme, offer a significant advantage: they eliminate the need to solve Riemann problems, thereby reducing computational complexity and making them more attractive for certain applications. For problems of the type described by (5.0.1), a staggered Lax-Friedrichs central scheme was analyzed in [116] and more recently in [118]. The second-order central scheme of Nessyahu and Tadmor, introduced in [137], can be viewed as an extension of the first-order LF scheme to the case where the flux function is continuous or, equivalently, where the coefficient function $k(x)$ in (5.0.1) is a constant. This scheme has been extensively studied further in the literature; see [24, 133, 138, 143, 144]. We also note that in [29], a modification of the NT scheme, namely the Kurganov-Tadmor [125] scheme, was adapted to handle the discontinuous flux case arising in the modeling of continuous separation of polydisperse mixtures, and its superior performance over a first-order scheme was computationally illustrated.

It is well known that for conservation laws with discontinuous flux function, solutions may fail to possess bounded total variation; see [3, 93]. In this case, as we described in Chapter 1, Section 1.2, the singular mapping technique is the commonly used framework for the convergence analysis of numerical schemes. However, applying the singular mapping technique to second-order schemes is challenging due to the difficulties in obtaining a time-continuity estimate in the absence of monotonicity of the scheme. This obstacle was overcome in [44] and [12] by imposing a limiter on the second-order scheme, which makes the scheme FTVD and consequently proving the time-continuity. Nevertheless, this limiting algorithm may seem somewhat tailored, lacking the straightforwardness of the standard slope-limiter methods. Since the proposed second-order scheme in our work is free from such additional limiters, it becomes necessary to explore an alternative framework for establishing convergence.

A key highlight of this work is the application of the theory of compensated compactness to establish the convergence of the proposed second-order scheme. The compensated compactness framework serves as a powerful tool for proving convergence of numerical schemes, particularly in cases where bounded variation (BV) estimates are unavailable. In this work, we employ the compensated compactness approach developed in Tartar's theory [79, 156]. This approach was previously employed in [116] to prove the convergence of the first-order Lax–Friedrichs scheme for scalar conservation laws with discontinuous coefficients; see also [82, 112] for related developments. The convergence proof relies on establishing a discrete maximum principle and deriving several key estimates to demonstrate the $W^{-1,2}_{\text{loc}}$ compactness of certain nonlinear terms involving the approximate solutions. For the second-order scheme, these estimates are significantly more delicate due to the presence of piecewise linear reconstruction in addition to the flux discontinuity, which introduces further analytical difficulties. A central step in the analysis is the derivation of a bound based on the concept of one-sided Lipschitz stability, as studied in [143]. This and several other carefully constructed estimates play a crucial role in the overall convergence result.

Further, to ensure the convergence of the proposed scheme to the entropy solution, following the approach in [12, 166], we incorporate a mesh-dependent term into the slope limiter. The core strategy involves writing the time-stepping in the proposed second-order scheme in a predictor-corrector form, where the predictor step employs a first-order accurate Lax-Friedrichs time-stepping, and the correction terms guarantee second-order accuracy. In broad terms, the mesh-dependent term in the modified slope limiter ensures that these correction terms vanish as the mesh size approaches zero. Building on this idea, we show that, as the mesh size tends to zero, our second-order scheme converges to the limit of the first-order Lax-Friedrichs scheme, which was shown to be the entropy solution in [116]. We note that the analysis presented in [12, 166] relies on bounded variation (BV) estimates of the approximate solutions. However, in the case of discontinuous flux, such BV estimates are not necessarily available; see [3, 93]. In this context, a key novelty of our approach lies in utilizing a weaker estimate, which we derive, to establish the entropy convergence.

We have organized the rest of this chapter as follows. Section 5.1 provides preliminary details related to the problem (5.0.1). In Section 5.2, we present the proposed second-order central scheme. The compensated compactness theory is outlined in Section 5.3. A maximum principle is proven for the proposed scheme in Section 5.4. Section 5.5 is dedicated to deriving a priori estimates. In Section 5.6, using the compensated compactness framework, we prove the convergence of the proposed scheme along a subsequence to a weak solution. Section 5.7 establishes the entropy convergence of the scheme, via the introduction of a mesh-size dependent term in the minmod slopes. Numerical results are presented in Section 5.8 and conclusions are drawn in Section 5.9. The derivation of certain crucial but lengthy technical estimates is deferred to Appendix C.1.

5.1 Preliminaries

5.1.1 Notations

In addition to the notations defined in Chapter 1, Section 1.0.0.1, we use the following notations throughout this chapter: For sequences $\{v_j\}_{j\in\mathbb{Z}}$ and $\{w_{j+\frac{1}{2}}\}_{j\in\mathbb{Z}}$ we denote $\Delta v_{j+\frac{1}{2}} := v_{j+1} - v_j$ and $\Delta w_j := w_{j+\frac{1}{2}} - w_{j-\frac{1}{2}}$, respectively, for $j \in \mathbb{Z}$. Also for $a \in \mathbb{R}$, denote $a_+ = \max\{a, 0\}$ and $a_- = \min\{a, 0\}$. For $a \in \mathbb{R}$, the greatest integer function is denoted by $\lfloor a \rfloor$, and the sign function by $\operatorname{sgn}(a)$. Finally, $\|\cdot\|_{BV}$ denotes the total variation semi-norm.

5.1.2 Hypotheses

We assume throughout this chapter that the initial data u_0 is such that

$$u_0 \in \mathrm{L}^\infty(\mathbb{R});\ u_0(x) \in [\underline{u}, \bar{u}] \text{ for a.e. } x \in \mathbb{R}, \tag{5.1.1}$$

for some $\underline{u}, \bar{u} \in \mathbb{R}$ such that $\underline{u} \le \bar{u}$. Further, we impose the following assumptions on the coefficient k and the flux function f.

(**H1**) For some $\underline{k}, \bar{k} \in \mathbb{R}$ such that $\underline{k} \le \bar{k}$ the function k satisfies

$$k \in (\mathrm{BV} \cap \mathrm{L}^\infty)(\mathbb{R}) \quad \text{and} \quad \underline{k} \le k(x) \le \bar{k} \text{ for a.e. } x \in \mathbb{R}. \tag{5.1.2}$$

(**H2**) For each fixed $k \in [\underline{k}, \bar{k}]$, the map $f(k, \cdot) : u \mapsto f(k, u) \in C^3[\underline{u}, \bar{u}]$ and is strictly convex. Moreover, there exists $\gamma_1, \gamma_2 \ge 0$ such that $0 < \gamma_1 \le f_{uu}(k, u) \le \gamma_2$ for all $u \in [\underline{u}, \bar{u}]$.

(**H3**) For each fixed $u \in [\underline{u}, \bar{u}]$, the map $f(\cdot, u) : k \mapsto f(k, u) \in C^2[\underline{k}, \bar{k}]$ with $\partial^2_{kk} f \equiv 0$.

(**H4**) The map $f_u : k \mapsto f_u(k, u) \in C^1[\underline{k}, \bar{k}]$.

(**H5**) The flux f satisfies $f(k_1, \bar{u}) = f(k_2, \bar{u})$ and $f(k_1, \underline{u}) = f(k_2, \underline{u})$ for all $k_1, k_2 \in [\underline{k}, \bar{k}]$. For multiplicative flux $f(k, u) = kg(u)$, this assumption reduces to $g(\underline{u}) = g(\bar{u}) = 0$.

(**H6**) The coefficient k is piecewise C^1 and is discontinuous only at finitely many points, say $D = \{x_1, x_2, \dots, x_M\}$.

(**H7**) Crossing condition: For any jump in the coefficient k with the corresponding left and right limits k_m^- and k_m^+ respectively,

$$f(k_m^+, u_1) - f(k_m^-, u_1) < 0 < f(k_m^+, u_2) - f(k_m^-, u_2) \implies u_1 < u_2,$$

for any states $u_1, u_2 \in [\underline{u}, \bar{u}]$.

Also, throughout this chapter we denote by $\kappa \in \mathbb{R}_+$ a positive constant such that

$$\lambda \|f_u\| \leq \kappa. \tag{5.1.3}$$

Remark 5.1.1. The analysis presented in this chapter is carried out under the assumption that the flux function satisfies the strict convexity condition stated in **H2**. However, we note that an analogous analysis can be performed for strictly concave fluxes as well.

5.1.3 Weak and entropy solutions

It is well established that the Cauchy problem (5.0.1) does not generally admit classical solutions, even when the coefficient k and the initial datum u_0 are smooth. Instead, solutions to (5.0.1) are interpreted in the following weak sense.

Definition 5.1.2. (Weak solution) A function $u \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be a weak solution of (5.0.1) if it satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} (u\phi_t + f(k(x), u)\phi_x) \,\mathrm{d}t \,\mathrm{d}x + \int_{\mathbb{R}} u_0 \phi(x, 0) \,\mathrm{d}x = 0, \tag{5.1.4}$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Weak solutions in the sense defined above need not be unique and an entropy condition needs to be specified to choose the relevant weak solution.

Definition 5.1.3. (Entropy solution) A function $u \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is called an entropy solution of (5.0.1) if for all $c \in \mathbb{R}$,

$$\begin{aligned}
&\int\int_{\mathbb{R}\times\mathbb{R}_+} (|u - c|\phi_t + \operatorname{sgn}(u - c)(f(k, u) - f(k, c))\phi_x) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |u_0 - c|\phi(x, 0) \,\mathrm{d}x \\
&\quad + \int\int_{(\mathbb{R}\backslash D)\times\mathbb{R}_+} |f(k(x), c)_x| \phi \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \sum_{m=1}^{M} \int_0^\infty |f(k_m^+, c) - f(k_m^-, c)| \phi(x_m, t) \,\mathrm{d}t \geq 0,
\end{aligned} \tag{5.1.5}$$

for all non-negative test functions $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$, where D is the set of discontinuities of k as given in **H6**.

Remark 5.1.4. The uniqueness of the entropy solution, as defined above, was established in [116] under the assumption of the crossing condition **H7**. However, in the two-flux case (1.2.2) with unimodal fluxes and a single flux crossing, this assumption can be omitted (see [46]). We note that our analysis can be extended to the setting of [46] also.

5.2 Second-order scheme

The spatial domain is discretized using a uniform mesh of size Δx into intervals of the form $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ where $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x$. For a fixed time step Δt, the time domain is discretized into points $t^n = n\Delta t$ for $n \in \{0, 1, \dots\}$. The ratio $\lambda = \frac{\Delta t}{\Delta x}$ is kept as a constant throughout. Further, the initial data u_0 is discretized as

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\,\mathrm{d}x \quad \text{for } j \in \mathbb{Z}.$$

Given the solution $\{u_j^n\}_{j\in\mathbb{Z}}$ at the time-level t^n, we compute the solutions $\{u_{j+\frac{1}{2}}^{n+1}\}_{j\in\mathbb{Z}}$, at the next time level t^{n+1} on the staggered grid following the approach outlined in [137]. We begin with a piecewise linear reconstruction of the cell averages $\{u_j^n\}$ utilizing a minmod slope limiter, as a step towards achieving second-order accuracy in space. At the time-level t^n, the solution in each cell $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ is then reconstructed as follows

$$\tilde{u}_j^n(x) = u_j^n + \frac{(x - x_j)}{\Delta x}\sigma_j^n, \quad x_{j+\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}, \tag{5.2.1}$$

where the slopes are given by

$$\sigma_j^n = \text{ minmod}\left((u_{j+1}^n - u_j^n), \frac{1}{2}(u_{j+1}^n - u_{j-1}^n), (u_j^n - u_{j-1}^n)\right), \tag{5.2.2}$$

the minmod function is defined by

$$\text{minmod}(a_1, \cdots, a_m) \coloneqq \begin{cases} \text{sgn}(a_1) \min\limits_{1\leq k\leq m}\{|a_k|\} & \text{if } \text{sgn}(a_1) = \cdots = \text{sgn}(a_m) \\ 0 & \text{otherwise.} \end{cases}$$

Now, a finite volume integration in the domain $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$ yields

$$\begin{aligned} \int_{x_j}^{x_{j+1}} u(x, t^{n+1})\,\mathrm{d}x &= \int_{x_j}^{x_{j+1}} u(x, t^n)\,\mathrm{d}x - \int_{t^n}^{t^{n+1}} f(k_{j+1}, u(x_{j+1}, t))\,\mathrm{d}t \\ &\quad + \int_{t^n}^{t^{n+1}} f(k_j, u(x_j, t))\,\mathrm{d}t \\ &\approx \int_{x_j}^{x_{j+\frac{1}{2}}} \tilde{u}_j^n(x, t^n)\,\mathrm{d}x + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} \tilde{u}_{j+1}^n(x, t^n)\,\mathrm{d}x \\ &\quad - \Delta t\left(f(k_{j+1}, u(x_{j+1}, t^{n+\frac{1}{2}})) - f(k_j, u(x_j, t^{n+\frac{1}{2}}))\right), \end{aligned} \tag{5.2.3}$$

using the midpoint quadrature rule in the time integration. Applying the definition (5.2.1) in (5.2.3), we now write the staggered second-order scheme as

$$u_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{8}(\sigma_{j+1}^n - \sigma_j^n) - \lambda\left(f(k_{j+1}, u_{j+1}^{n+1/2}) - f(k_j, u_j^{n+1/2})\right), \tag{5.2.4}$$

where the mid-time step values are computed as

$$u_j^{n+\frac{1}{2}} = u_j^n - \frac{\Delta t}{2\Delta x} f_u(k_j, u_j^n)\sigma_j^n, \quad j \in \mathbb{Z}. \tag{5.2.5}$$

Alternatively, we can also express the scheme (5.2.4) as

$$u_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \lambda \left(g(k_{j+1}, u_{j+1}^n) - g(k_j, u_j^n) \right), \tag{5.2.6}$$

where g is given by

$$g(k_j, u_j^n) := f(k_j, u_j^{n+\frac{1}{2}}) + \frac{1}{8\lambda}\sigma_j^n. \tag{5.2.7}$$

For a fixed mesh size Δx, the piecewise constant approximate solution and the discretized discontinuous coefficient k are represented by the pair

$$\begin{aligned} &(u_\Delta(x,t), k_\Delta(x,t)) \\ &:= \begin{cases} (u_j^n, k_j) & \text{if } n \text{ is even and } (x,t) \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1}), \\ (u_{j+\frac{1}{2}}^n, k_{j+\frac{1}{2}}) & \text{if } n \text{ is odd and } (x,t) \in [x_j, x_{j+1}) \times [t^n, t^{n+1}), \end{cases} \end{aligned} \tag{5.2.8}$$

where $n \in \mathbb{N} \cup \{0\}$ and

$$k_j := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} k(x)\,\mathrm{d}x \quad \text{and} \quad k_{j+\frac{1}{2}} := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k(x)\,\mathrm{d}x. \tag{5.2.9}$$

Remark 5.2.1. When the slopes $\sigma_j = 0$ for all j, the scheme reduces to the first-order Lax-Friedrichs scheme

$$u_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \lambda \left(f(k_{j+1}, u_{j+1}^n) - f(k_j, u_j^n) \right), \tag{5.2.10}$$

considered in [116]. In other words, the scheme (5.2.4) is a second-order extension of the scheme (5.2.10).

5.3 Compensated compactness

The main ingredient of the compensated compactness framework is a theorem given below, the proof of which can be found in [116]. In the sequel, we will show that the proposed scheme (5.2.4) meets the requirements outlined in this theorem. This will ensure the existence of a convergent subsequence of approximate solutions generated from the scheme (5.2.4).

Theorem 5.3.1 (Compensated compactness theorem)**.** *Assume that the hypotheses **H1**-**H5** hold true. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of measurable functions defined on $\mathbb{R} \times \mathbb{R}^+$ that satisfies the following two conditions:*

1. *There exist $a, b \in \mathbb{R}$ with $a < b$, both independent of ε, such that*

$$a \leq u^\varepsilon(x,t) \leq b \text{ for a.e. } (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

2. *The two sequences*

$$\{S_1(u^\varepsilon)_t + Q_1(k(x), u^\varepsilon)_x\}_{\varepsilon>0} \quad \text{and} \quad \{S_2(k(x), u^\varepsilon)_t + Q_2(k(x), u^\varepsilon)_x\}_{\varepsilon>0},$$

belong to a compact subset of $\mathrm{W}^{-1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$, where

$$\begin{aligned} S_1(u) &:= u - c, \quad Q_1(k,u) := f(k,u) - f(k,c), \\ S_2(k,u) &:= f(k,u) - f(k,c) \quad \text{and} \quad Q_2(k,u) := \int_c^u (f_u(k,\xi))^2 \, \mathrm{d}\xi, \end{aligned} \tag{5.3.1}$$

for any $c \in \mathbb{R}$.

Then, there exists a subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$ that converges pointwise a.e. to a function $u \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{R}^+)$.

To establish the $\mathrm{W}^{-1,2}_{\mathrm{loc}}$ compactness required by Theorem 5.3.1, we will make use of the following interpolation result as well. For a proof of this result, see [78].

Lemma 5.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Let q and r be a pair of constants satisfying $1 < q < 2 < r < \infty$. If A is compact subset of $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\Omega)$ and B is a bounded subset of $\mathrm{W}^{-1,r}_{\mathrm{loc}}(\Omega)$, then*

$$A \cap B \text{ is compact in } \mathrm{W}^{-1,2}_{\mathrm{loc}}(\Omega).$$

5.4 Maximum principle and L^∞-stability

This section establishes that the approximate solutions u_Δ, as defined in (5.2.8), satisfy a global maximum principle, thereby yielding an L^∞-estimate as well.

Theorem 5.4.1. *Let the initial datum $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ with $\underline{u} \leq u_0(x) \leq \bar{u}$, for all $x \in \mathbb{R}$. Then, under the CFL condition*

$$\lambda \|f_u\| \leq \kappa \leq \frac{\sqrt{2}-1}{2}, \tag{5.4.1}$$

*and hypotheses **H1-H5**, the approximate solution u_Δ (5.2.8) obtained from the scheme* (5.2.4) *satisfies the global maximum principle*

$$\underline{u} \leq u_\Delta(x,t) \leq \bar{u}, \tag{5.4.2}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Consequently, the approximate solutions u_Δ are L^∞-stable, i.e.,

$$\|u_\Delta\| \leq C_{u_0} := \max\{|\underline{u}|, |\bar{u}|\}. \tag{5.4.3}$$

Proof. We use the principle of mathematical induction to prove this result. By the assumption on the initial datum u_0, the result holds for $n = 0$. For $n \geq 0$, suppose $u_j^n \in [\underline{u}, \bar{u}]$, for all $j \in \mathbb{Z}$. We will now prove that $u_{j+\frac{1}{2}}^{n+1} \in [\underline{u}, \bar{u}]$ for all $j \in \mathbb{Z}$. Adding and subtracting the term $f(k_{j+1}, \bar{u})$, and using the hypothesis **H5** (i.e., $f(k_{j+1}, \bar{u}) = f(k_j, \bar{u})$), the difference $f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_j, u_j^{n+\frac{1}{2}})$ can be expressed as

$$\begin{aligned}
f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_j, u_j^{n+\frac{1}{2}}) &= f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_{j+1}, \bar{u}) + f(k_j, \bar{u}) - f(k_j, u_j^{n+\frac{1}{2}}) \\
&= f_u(k_{j+1}, \tilde{\zeta}_{j+1})(u_{j+1}^{n+\frac{1}{2}} - \bar{u}) + f_u(k_j, \tilde{\zeta}_j)(\bar{u} - u_j^{n+\frac{1}{2}}) \\
&= f_u(k_{j+1}, \tilde{\zeta}_{j+1})(u_{j+1}^{n} - \bar{u}) + f_u(k_j, \tilde{\zeta}_j)(\bar{u} - u_j^{n}) \\
&\quad - \frac{1}{2}\lambda f_u(k_{j+1}, u_{j+1}^n) f_u(k_{j+1}, \tilde{\zeta}_{j+1})\sigma_{j+1}^n \\
&\quad + \frac{1}{2}\lambda f_u(k_j, u_j^n) f_u(k_j, \tilde{\zeta}_j)\sigma_j^n,
\end{aligned} \tag{5.4.4}$$

where $\tilde{\zeta}_{j+1} \in \mathcal{I}(u_{j+1}^{n+\frac{1}{2}}, \bar{u})$ and $\tilde{\zeta}_j \in \mathcal{I}(u_j^{n+\frac{1}{2}}, \bar{u})$. From the expression (5.4.4), we get the estimate

$$\left|f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_j, u_j^{n+\frac{1}{2}})\right| \leq \|f_u\| \left(2\bar{u} - u_j^n - u_{j+1}^n\right) + \lambda(\|f_u\|)^2 |u_{j+1}^n - u_j^n|. \tag{5.4.5}$$

From the CFL condition (5.4.1), we obtain $\frac{1}{4} + \kappa^2 \leq \frac{1}{2} - \kappa$. This inequality, when combined with the estimate (5.4.5) applied on the scheme (5.2.4), and (5.1.3), yields

$$\begin{aligned}
u_{j+\frac{1}{2}}^{n+1} &\leq \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{4}|u_{j+1}^n - u_j^n| + \lambda\|f_u\| \left(2\bar{u} - u_j^n - u_{j+1}^n\right) + \lambda^2\|f_u\|^2 |u_{j+1}^n - u_j^n| \\
&\leq \left(\frac{1}{2} - \kappa\right)(u_j^n + u_{j+1}^n) + \left(\frac{1}{4} + \kappa^2\right)|u_{j+1}^n - u_j^n| + 2\kappa\bar{u} \\
&\leq \left(\frac{1}{2} - \kappa\right) 2\max\{u_j^n, u_{j+1}^n\} + 2\kappa\bar{u} = (1 - 2\kappa)\max\{u_j^n, u_{j+1}^n\} + 2\kappa\bar{u} \leq \bar{u}.
\end{aligned} \tag{5.4.6}$$

By the CFL condition (5.4.1), $2\kappa \in [0, 1]$ and hence the term $(1 - 2\kappa)\max\{u_j^n, u_{j+1}^n\}$ $+ 2\kappa\bar{u}$ is a convex combination of points in $[\underline{u}, \bar{u}]$. As a result, the last inequality in the previous equation holds true. Similar arguments as in (5.4.4), by adding and subtracting the term $f(k_{j+1}, \underline{u})$ in $f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_j, u_j^{n+\frac{1}{2}})$, gives another estimate

$$\left|f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - f(k_j, u_j^{n+\frac{1}{2}})\right| \leq \|f_u\| \left(u_j^n + u_{j+1}^n - 2\underline{u}\right) + \lambda(\|f_u\|)^2 |u_{j+1}^n - u_j^n|,$$

which subsequently yields the lower bound

$$\begin{aligned}
u_{j+\frac{1}{2}}^{n+1} &\geq \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{4}|u_{j+1}^n - u_j^n| - \lambda\|f_u\| \left(u_j^n + u_{j+1}^n - 2\underline{u}\right) - \lambda^2\|f_u\|^2 |u_{j+1}^n - u_j^n| \\
&\geq \left(\frac{1}{2} - \kappa\right)(u_j^n + u_{j+1}^n) - \left(\frac{1}{4} + \kappa^2\right)|u_{j+1}^n - u_j^n| + 2\kappa\underline{u} \\
&\geq \left(\frac{1}{2} - \kappa\right)\left((u_j^n + u_{j+1}^n) - |u_{j+1}^n - u_j^n|\right) + 2\kappa\underline{u}
\end{aligned} \tag{5.4.7}$$

$$= (1-2\kappa)\min\{u_j^n, u_{j+1}^n\} + 2\kappa \underline{u} \geq \underline{u}.$$

The expressions (5.4.6) and (5.4.7) together yield the maximum principle (5.4.2). $\square$

5.5 A priori estimates

In this section, we derive certain essential a priori estimates on the approximate solutions obtained from the second-order scheme (5.2.4). These estimates form the foundation for the convergence analysis presented in Section 5.6.

First, we focus on estimating the term $\sum_{j\in\mathbb{Z}}(u_{j+1}^n - u_j^n)_+^3$, where $\{u_j^n\}_{j\in\mathbb{Z}}$ are the solutions derived from the second-order scheme (5.2.4). The details of this estimate are outlined in the following lemma. The proof is lengthy and is provided in Appendix C.1.

Lemma 5.5.1. *Assume $\{u_j^n\}_{j\in\mathbb{Z}}$ is such that $|u_j| \leq C_{u_0}$, $\forall j \in \mathbb{Z}$, where C_{u_0} is as in* (5.4.3). *Let $\{u_{j+\frac{1}{2}}^{n+1}\}_{j\in\mathbb{Z}}$ be obtained from $\{u_j^n\}_{j\in\mathbb{Z}}$ by applying the time-update formula* (5.2.4). *Under the CFL condition*

$$\lambda\|f_u\| \leq \kappa \leq \min\left\{\frac{\gamma_1}{7500\gamma_2}, \frac{1}{4000}\right\}, \tag{5.5.1}$$

the solution $\{u_{j+\frac{1}{2}}^{n+1}\}_{j\in\mathbb{Z}}$ satisfies the estimate

$$\sum_{j\in\mathbb{Z}} \left(\Delta u_j^{n+1}\right)_+^2 \leq \sum_{j\in\mathbb{Z}}(\Delta u_{j-\frac{1}{2}}^n)_+^2 - \frac{1}{500}\lambda\gamma_1 \sum_{j\in\mathbb{Z}}(\Delta u_{j-\frac{1}{2}}^n)_+^3 + \Psi\|k\|_{BV}, \tag{5.5.2}$$

for all $n \geq 0$, where

$$\begin{aligned}\Psi := {} & 72\lambda^2(C_{u_0})^2\|f_{uk}\| + 114\lambda(C_{u_0})^2\|f_{ku}\| + \left(708(C_{u_0})^2 + 48\lambda\|f_u\|\right)\lambda^2\|f_u\|\|f_{uk}\| \\ & + \left(48\lambda^2 C_{u_0}\|k\| + 132\lambda^2(C_{u_0})^2\gamma_2\|f_u\| + 64\lambda\|f_k\|\|k\| + 88C_{u_0}\right)\lambda\|f_k\|.\end{aligned}$$

Next, we proceed to obtain a cubic estimate on the spatial differences of the approximate solutions.

Lemma 5.5.2. *(Cubic estimate) Let the initial datum $u_0 \in (\mathrm{L}^\infty \cap \mathrm{BV})(\mathbb{R})$ be such that $\|u_0\| \leq C_{u_0}$. For any fixed $T > 0, X > 0$, define $N := \lfloor T/\Delta t\rfloor + 1$ and $J := \lfloor X/\Delta x\rfloor + 1$. Then under the CFL condition*

$$\lambda\|f_u\| \leq \kappa \leq \min\left\{\frac{\gamma_1}{7500\gamma_2}, \frac{1}{4000}, \frac{7}{85+16C_{u_0}}, \frac{\gamma_1}{\gamma_2\chi}\right\}, \tag{5.5.3}$$

with $\chi := 228 + 13C_{u_0} + 174C_{u_0}\gamma_2 + 12(C_{u_0})^2\gamma_2$, the approximate solutions generated by the second-order scheme (5.2.4) *satisfy the uniform bound*

$$\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J \\ j+\frac{n}{2}\in\mathbb{Z}}} |\Delta u_{j+\frac{1}{2}}^n|^3 \leq C(X,T), \tag{5.5.4}$$

for a constant $C(X,T)$ independent of Δx.

Proof. We need to define a linear function $\bar{g}_{j+\frac{1}{2}}$ which interpolates $g(k_j, u_j^n)$ and $g(k_{j+1}, u_{j+1}^n)$ as follows:

$$\bar{g}_{j+\frac{1}{2}}(u) := g(k_j, u_j^n) + \frac{\Delta g_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}(u - u_j^n) \text{ for } u \in \mathcal{I}(u_j^n, u_{j+1}^n),$$

where $\Delta g_{j+\frac{1}{2}}^n := g(k_{j+1}, u_{j+1}^n) - g(k_j, u_j^n)$. Further, we consider functions $S, Q : [\underline{k}, \bar{k}] \times [\underline{u}, \bar{u}] \to \mathbb{R}$, with the property that $\partial_u Q = \partial_u S \partial_u f$. Now, we define a quantity $E_{j+\frac{1}{2}}^n$ associated with the pair (S, Q) as

$$E_{j+\frac{1}{2}}^n := S(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n+1}) - \frac{1}{2}\left(S(k_j, u_j^n) + S(k_{j+1}, u_{j+1}^n)\right) + \lambda\left(Q(k_{j+1}, u_{j+1}^n) - Q(k_j, u_j^n)\right). \tag{5.5.5}$$

Our objective now is to reformulate $E_{j+\frac{1}{2}}^n$ in a suitable form, and use that to obtain the desired estimate. We begin with rewriting $E_{j+\frac{1}{2}}^n$ by adding and subtracting suitable terms, as

$$\begin{aligned} E_{j+\frac{1}{2}}^n = S(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n+1}) &- \frac{1}{2}\Big(S(k_{j+\frac{1}{2}}, u_j^n) + S(k_{j+\frac{1}{2}}, u_{j+1}^n)\Big) \\ &+ \lambda\Big(Q(k_{j+\frac{1}{2}}, u_{j+1}^n) - Q(k_{j+\frac{1}{2}}, u_j^n)\Big) + R_1 + R_2, \end{aligned} \tag{5.5.6}$$

where

$$\begin{aligned} R_1 &:= \frac{1}{2}\Big(S(k_{j+\frac{1}{2}}, u_j^n) + S(k_{j+\frac{1}{2}}, u_{j+1}^n)\Big) - \frac{1}{2}\left(S(k_j, u_j^n) + S(k_{j+1}, u_{j+1}^n)\right) \\ &= \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}), \\ R_2 &:= \lambda\Big[Q(k_{j+1}, u_{j+1}^n) - Q(k_j, u_j^n) - \Big(Q(k_{j+\frac{1}{2}}, u_{j+1}^n) - Q(k_{j+\frac{1}{2}}, u_j^n)\Big)\Big] \\ &= \mathcal{O}(k_j - k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+1} - k_{j+\frac{1}{2}}). \end{aligned} \tag{5.5.7}$$

Now, we define parametrized functions for $s \in [0, 1]$ as

$$\begin{aligned} u(s) &:= s u_j^n + (1-s) u_{j+1}^n, \\ u_{j+\frac{1}{2}}(s) &:= \frac{1}{2}\left(u(s) + u_{j+1}^n\right) - \lambda(g(k_{j+1}, u_{j+1}^n) - \bar{g}(u(s))), \\ u(r, s) &:= r u(s) + (1-r) u_{j+1}^n \quad \text{and} \\ u_{j+\frac{1}{2}}(r, s) &:= \frac{1}{2}\left(u(s) + u(r, s)\right) - \lambda(\bar{g}(u(r, s)) - \bar{g}(u(s))). \end{aligned} \tag{5.5.8}$$

These functions satisfy the properties

$$\begin{aligned} &u(0) = u_{j+1}^n, \quad u(1) = u_j^n, \quad u_{j+\frac{1}{2}}(0) = u_{j+1}^n, \quad u_{j+\frac{1}{2}}(1) = u_{j+\frac{1}{2}}^{n+1}, \\ &u(0, s) = u_{j+1}^n, \quad u(1, s) := u(s), \quad u_{j+\frac{1}{2}}(0, s) = u_{j+\frac{1}{2}}(s), \quad u_{j+\frac{1}{2}}(1, s) = u(s). \end{aligned} \tag{5.5.9}$$

Next, using (5.5.8), we write

$$\begin{aligned} & S(k_{j+\frac{1}{2}}, u^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(k_{j+\frac{1}{2}}, u^n_j) + S(k_{j+\frac{1}{2}}, u^n_{j+1})\right) \\ & = \int_0^1 S_u(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}(s))\left(\frac{1}{2} + \lambda \bar{g}'_{j+\frac{1}{2}}(u(s))\right) u'(s)\,\mathrm{d}s - \frac{1}{2}\int_0^1 S_u(k_{j+\frac{1}{2}}, u(s)) u'(s)\,\mathrm{d}s. \end{aligned} \tag{5.5.10}$$

Also, if $\tilde{Q}$ is defined such that $\tilde{Q}_u = S_u \bar{g}'_{j+\frac{1}{2}} = S_u \dfrac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}$, then adding and subtracting appropriate terms, we get

$$\begin{aligned} & \lambda\left(Q(k_{j+\frac{1}{2}}, u^n_{j+1} - Q(k_{j+\frac{1}{2}}, u^n_j))\right) \\ & = \lambda(\tilde{Q}(k_{j+\frac{1}{2}}, u^n_{j+1}) - \tilde{Q}(k_{j+\frac{1}{2}}, u^n_j)) \\ & \quad + \lambda\left[\left(Q(k_{j+\frac{1}{2}}, u^n_{j+1}) - \tilde{Q}(k_{j+\frac{1}{2}}, u^n_{j+1})\right) - \left(Q(k_{j+\frac{1}{2}}, u^n_j) - \tilde{Q}(k_{j+\frac{1}{2}}, u^n_j)\right)\right] \\ & = -\lambda \int_0^1 S_u(k_{j+\frac{1}{2}}, u(s)) \bar{g}'_{j+\frac{1}{2}}(u(s)) u'(s)\,\mathrm{d}s \\ & \quad - \lambda \int_0^1 S_u(k_{j+\frac{1}{2}}, u(s))\left(f_u(k_{j+\frac{1}{2}}, u(s)) - \bar{g}'_{j+\frac{1}{2}}(u(s))\right) u'(s)\,\mathrm{d}s. \end{aligned} \tag{5.5.11}$$

Now, in view of (5.5.10) and (5.5.11), and rearranging the terms, we write

$$E^n_{j+\frac{1}{2}} = I + J + R^n_1 + R^n_2, \tag{5.5.12}$$

where

$$\begin{aligned} I &:= \int_0^1 \left(S_u(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}(s)) - S_u(k_{j+\frac{1}{2}}, u(s))\right)\left(\frac{1}{2} + \lambda \bar{g}'_{j+\frac{1}{2}}(u(s))\right) u'(s)\,\mathrm{d}s, \\ J &:= -\lambda \int_0^1 S_u(k_{j+\frac{1}{2}}, u(s))\left(f_u(k_{j+\frac{1}{2}}, u(s)) - \bar{g}'_{j+\frac{1}{2}}(u(s))\right) u'(s)\,\mathrm{d}s. \end{aligned} \tag{5.5.13}$$

Using integration by parts, J can be simplified as

$$J = \bar{J} - \lambda\left[S_u(k_{j+\frac{1}{2}}, u(s))\left(f(k_{j+\frac{1}{2}}, u(s)) - \bar{g}_{j+\frac{1}{2}}(u(s))\right)\right]_0^1, \tag{5.5.14}$$

where $\bar{J} := \lambda \int_0^1 S_{uu}(k_{j+\frac{1}{2}}, u(s))\left(f(k_{j+\frac{1}{2}}, u(s)) - \bar{g}_{j+\frac{1}{2}}(u(s))\right) u'(s)\,\mathrm{d}s$. Further, by adding and subtracting appropriate terms in (5.5.14), we obtain

$$\begin{aligned} J = \bar{J} - \lambda & \left[S_u(k_j, u^n_j)(f(k_j, u^n_j) - g(k_j, u^n_j)) \right. \\ & \left. - S_u(k_{j+1}, u^n_{j+1})(f(k_{j+1}, u^n_{j+1}) - g(k_{j+1}, u^n_{j+1}))\right] + \tilde{R}^n_1 + \tilde{R}^n_2, \end{aligned} \tag{5.5.15}$$

where

$$\begin{aligned} \tilde{R}^n_1 := & -\lambda\left[S_u(k_{j+\frac{1}{2}}, u^n_j) f(k_{j+\frac{1}{2}}, u^n_j) - S_u(k_j, u_j) f(k_j, u^n_j)\right] \\ & + \lambda\left[S_u(k_{j+\frac{1}{2}}, u^n_{j+1}) f(k_{j+\frac{1}{2}}, u^n_{j+1}) - S_u(k_{j+1}, u^n_{j+1}) f(k_{j+1}, u^n_{j+1})\right] \end{aligned} \tag{5.5.16}$$

$$
\begin{aligned}
&= \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}),\\
\tilde{R}_2^n &:= -\lambda \left[\left(S_u(k_j, u_j^n) - S_u(k_{j+\frac{1}{2}}, u_j^n) \right) g(k_j, u_j^n) \right]\\
&\quad + \lambda \left[\left(S_u(k_{j+1}, u_{j+1}^n) - S_u(k_{j+\frac{1}{2}}, u_{j+1}^n) \right) g(k_{j+1}, u_{j+1}^n) \right]\\
&= \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+1} - k_{j+\frac{1}{2}}).
\end{aligned}
$$

At this stage, equating the right hand sides of (5.5.5) and (5.5.12) and using (5.5.15), we may write

$$
\bar{E}_{j+\frac{1}{2}}^n = I + \bar{J} + R_1^n + R_2^n + \tilde{R}_1^n + \tilde{R}_2^n, \tag{5.5.17}
$$

where we define

$$
\begin{aligned}
\bar{E}_{j+\frac{1}{2}}^n &:= S(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n+1}) - \frac{1}{2}\left(S(k_j, u_j^n) + S(k_{j+1}, u_{j+1}^n) \right) \\
&\quad + \lambda \left(G(k_{j+1}, u_{j+1}^n) - G(k_j, u_j^n) \right)), \\
G(k_j, u_j) &:= Q(k_j, u_j^n) - S_u(k_j, u_j^n) \left(f(k_j, u_j^n) - g(k_j, u_j^n) \right).
\end{aligned} \tag{5.5.18}
$$

Plugging in the derivative $\frac{\partial}{\partial r} u_{j+\frac{1}{2}}(r,s) = -s\Delta u_{j+\frac{1}{2}}^n \left(\frac{1}{2} - \lambda \bar{g}'_{j+\frac{1}{2}}(u(r,s)) \right)$ in (5.5.13), the term I can be represented as

$$
I = -(\Delta u_{j+\frac{1}{2}}^n)^2 \tilde{I}, \quad \text{where} \tag{5.5.19}
$$

$\tilde{I} := \int_0^1 \int_0^1 s S_{uu}(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}(r,s)) \left(\frac{1}{2} - \lambda \bar{g}'_{j+\frac{1}{2}}(u(r,s)) \right) \left(\frac{1}{2} + \lambda \bar{g}'_{j+\frac{1}{2}}(u(s)) \right) \mathrm{d}s \, \mathrm{d}r.$

Now, we set $S(k,u) = \dfrac{u^2}{2}$, and re-work on (5.5.17), specifically focusing on I and $\bar{J}$. Noting $S_{uu} = 1$, performing a change of variable $z = u(s)$ and subsequently applying the trapezoidal rule, the term $\bar{J}$ simplifies to

$$
\begin{aligned}
\bar{J} &= -\lambda \int_{u_j}^{u_{j+1}} f(k_{j+\frac{1}{2}}, z)\,\mathrm{d}z + \lambda \int_{u_j}^{u_{j+1}} \bar{g}_{j+\frac{1}{2}}(z)\,\mathrm{d}z \\
&= -\frac{1}{2}\lambda \Delta u_{j+\frac{1}{2}} \left[f(k_{j+\frac{1}{2}}, u_j^n) - g(k_j, u_j^n) + f(k_{j+\frac{1}{2}}, u_{j+1}^n) - g(k_{j+1}, u_{j+1}^n) \right] \\
&\quad + \frac{\lambda}{12} (\Delta u_{j+\frac{1}{2}}^n)^3 f_{uu}(k_{j+\frac{1}{2}}, \zeta_1),
\end{aligned} \tag{5.5.20}
$$

for some $\zeta_1 \in \mathcal{I}(u_j, u_{j+1})$. Recalling the definition $g(k_j, u_j^n) := f(k_j, u_j^{n+\frac{1}{2}}) + \frac{1}{8\lambda}\sigma_j^n$ and using Taylor series expansions in the second variable of g, we obtain

$$
\begin{aligned}
g(k_j, u_j^n) &= f(k_j, u_j^n) - \frac{\lambda}{2}(a_j^n)^2 \sigma_j^n + \frac{1}{8}(\lambda a_j^n \sigma_j^n)^2 f_{uu}(k_j, \zeta_2) + \frac{1}{8\lambda}\sigma_j^n, \\
g(k_{j+1}, u_{j+1}^n) &= f(k_{j+1}, u_{j+1}^n) - \frac{\lambda}{2}(a_{j+1}^n)^2 \sigma_{j+1}^n + \frac{1}{8}(\lambda a_{j+1}^n \sigma_{j+1}^n)^2 f_{uu}(k_{j+1}, \zeta_3) + \frac{1}{8\lambda}\sigma_{j+1}^n,
\end{aligned} \tag{5.5.21}
$$

for $\zeta_2 \in \mathcal{I}(u_j^n, u_j^{n+\frac{1}{2}})$ and $\zeta_3 \in \mathcal{I}(u_{j+1}^n, u_{j+1}^{n+\frac{1}{2}})$, where $a_j^n = f_u(k_j, u_j^n)$, as in (C.1.4). By adding and subtracting the terms $f(k_j, u_j^n)$ and $f(k_{j+1}, u_{j+1}^n)$, respectively, and using

(5.5.21) we obtain

$$
\begin{aligned}
f(k_{j+\frac{1}{2}}, u_j^n) - g(k_j, u_j^n) &= \frac{\lambda}{2}(a_j^n)^2\sigma_j^n - \frac{1}{8}(\lambda a_j^n\sigma_j^n)^2 f_{uu}(k_j, \zeta_2) - \frac{1}{8\lambda}\sigma_j^n \\
&\quad + \mathcal{O}(k_{j+\frac{1}{2}} - k_j), \\
f(k_{j+\frac{1}{2}}, u_{j+1}^n) - g(k_{j+1}, u_{j+1}^n) &= \frac{\lambda}{2}(a_{j+1}^n)^2\sigma_{j+1}^n - \frac{1}{8}(\lambda a_{j+1}^n\sigma_{j+1}^n)^2 f_{uu}(k_{j+1}, \zeta_3) \\
&\quad - \frac{1}{8\lambda}\sigma_{j+1}^n + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}),
\end{aligned}
\tag{5.5.22}
$$

Further, substituting (5.5.22) in (5.5.20) yields

$$
\begin{aligned}
\bar{J} &= -\frac{1}{2}\lambda\Delta u_{j+\frac{1}{2}}^n \left[\frac{\lambda}{2}(a_j^n)^2\sigma_j^n + \frac{\lambda}{2}(a_{j+1}^n)^2\sigma_{j+1}^n - \frac{1}{8\lambda}\sigma_j^n - \frac{1}{8\lambda}\sigma_{j+1}^n\right] \\
&\quad + \frac{1}{16}\lambda(\Delta u_{j+\frac{1}{2}}^n)^3 \left[\left(\lambda a_{j+1}^n\left(\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right)^2 + \left(\lambda a_j^n\left(\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right)^2\right. \\
&\quad \left. + \frac{4}{3} f_{uu}(k_{j+\frac{1}{2}}, \zeta_1)\right] + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}), \\
&= \frac{1}{16}\Delta u_{j+\frac{1}{2}}^n\mathcal{A}_1 + \frac{1}{16}\lambda(\Delta u_{j+\frac{1}{2}}^n)^3\mathcal{A}_2 + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}),
\end{aligned}
\tag{5.5.23}
$$

where we define

$$
\begin{aligned}
\mathcal{A}_1 &:= \sigma_j^n + \sigma_{j+1}^n - 4\lambda^2(a_j^n)^2\sigma_j^n - 4\lambda^2(a_{j+1}^n)^2\sigma_{j+1}^n, \\
\mathcal{A}_2 &:= \left(\lambda a_{j+1}^n\left(\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right)^2 + \left(\lambda a_j^n\left(\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right)^2 + \frac{4}{3} f_{uu}(k_{j+\frac{1}{2}}, \zeta_1).
\end{aligned}
\tag{5.5.24}
$$

Now, we concentrate on the terms $\mathcal{A}_1$ and $\mathcal{A}_2$. The hypothesis **H2** along with (5.1.3) give us

$$
\mathcal{A}_2 \geq \frac{4}{3}\gamma_1 \quad \text{and} \quad |\mathcal{A}_2| \leq 2\kappa^2 + \frac{4}{3}\gamma_2.
\tag{5.5.25}
$$

Next, introducing the notations

$$
\tilde{k}_{j+\frac{1}{2}} := \frac{k_j + k_{j+1}}{2}, \quad \tilde{u}_{j+\frac{1}{2}}^n := \frac{u_j^n + u_{j+1}^n}{2}, \quad \text{and} \quad \tilde{a}_{j+\frac{1}{2}}^n := f_u(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}_{j+\frac{1}{2}}^n),
\tag{5.5.26}
$$

and using the Taylor expansion of f_u about the point $(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}_{j+\frac{1}{2}}^n)$, we obtain

$$
\begin{aligned}
a_j^n &= \tilde{a}_{j+\frac{1}{2}}^n - \frac{1}{2}(k_{j+1} - k_j) f_{uk}(c_4, \zeta_4) - \frac{1}{2}(u_{j+1}^n - u_j^n) f_{uu}(c_4, \zeta_4), \\
a_{j+1}^n &= \tilde{a}_{j+\frac{1}{2}}^n + \frac{1}{2}(k_{j+1} - k_j) f_{uk}(c_5, \zeta_5) + \frac{1}{2}(u_{j+1}^n - u_j^n) f_{uu}(c_5, \zeta_5).
\end{aligned}
\tag{5.5.27}
$$

where $c_4 \in \mathcal{I}(k_j, \tilde{k}_{j+\frac{1}{2}})$, $c_5 \in \mathcal{I}(k_{j+1}, \tilde{k}_{j+\frac{1}{2}})$, $\zeta_4 \in \mathcal{I}(u_j^n, \tilde{u}_{j+\frac{1}{2}}^n)$ and $\zeta_5 \in \mathcal{I}(u_{j+1}^n, \tilde{u}_{j+\frac{1}{2}}^n)$. From these, we obtain

$$
\begin{aligned}
(a_j^n)^2 &= (\tilde{a}_{j+\frac{1}{2}}^n)^2 + \frac{1}{4}(\Delta u_{j+\frac{1}{2}}^n)^2 (f_{uu}(c_4, \zeta_4))^2 - \tilde{a}_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_4, \zeta_4) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \\
(a_{j+1}^n)^2 &= (\tilde{a}_{j+\frac{1}{2}}^n)^2 + \frac{1}{4}(\Delta u_{j+\frac{1}{2}}^n)^2 (f_{uu}(c_5, \zeta_5))^2 + \tilde{a}_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_5, \zeta_5) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}),
\end{aligned}
\tag{5.5.28}
$$

Further, using (5.5.28) and the notation $\beta := \lambda \tilde{a}^n_{j+\frac{1}{2}}$, the term $\mathcal{A}_1$ in (5.5.24) can be reformulated as

$$\mathcal{A}_1 = \left(1 - 4\beta^2\right)(\sigma^n_j + \sigma^n_{j+1}) - \lambda(\Delta u^n_{j+\frac{1}{2}})^2 \mathcal{A}_3 + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \tag{5.5.29}$$

where

$$\begin{aligned} \mathcal{A}_3 := {} & 4\beta \frac{\sigma^n_{j+1}}{\Delta u^n_{j+\frac{1}{2}}} f_{uu}(c_5, \zeta_5) - 4\beta \frac{\sigma^n_j}{\Delta u^n_{j+\frac{1}{2}}} f_{uu}(c_4, \zeta_4) \\ & + \lambda (f_{uu}(c_4, \zeta_4))^2 \sigma^n_j + \lambda (f_{uu}(c_5, \zeta_5))^2 \sigma^n_{j+1}. \end{aligned} \tag{5.5.30}$$

Replacing the term $\mathcal{A}_1$ in (5.5.23) with its expression (5.5.29), $\bar{J}$ can be rewritten as

$$\begin{aligned} \bar{J} = {} & \frac{1}{16}(\Delta u^n_{j+\frac{1}{2}})^2 \left(1 - 4\beta^2\right) \frac{(\sigma^n_j + \sigma^n_{j+1})}{\Delta u^n_{j+\frac{1}{2}}} + \frac{1}{16}\lambda(\Delta u^n_{j+\frac{1}{2}})^3 (\mathcal{A}_2 - \mathcal{A}_3) \\ & + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \end{aligned} \tag{5.5.31}$$

Next, we focus on the term I in (5.5.19). The choice $S(k,u) = \frac{u^2}{2}$ simplifies I as

$$I = -\frac{1}{8}(\Delta u^n_{j+\frac{1}{2}})^2 \left(1 - \left(2\lambda \frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2\right). \tag{5.5.32}$$

Here, using (5.5.21), the term $\dfrac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}$ in (5.5.32) can be expressed as follows

$$\frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}} = L + M + N, \tag{5.5.33}$$

where

$$\begin{aligned} L & := \frac{f(k_{j+1}, u^n_{j+1}) - f(k_j, u^n_j)}{\Delta u^n_{j+\frac{1}{2}}}, \\ M & := -\frac{\left(\sigma^n_{j+1}\left(\frac{\lambda}{2}(a^n_{j+1})^2 - \frac{1}{8\lambda}\right) - \sigma^n_j\left(\frac{\lambda}{2}(a^n_j)^2 - \frac{1}{8\lambda}\right)\right)}{\Delta u^n_{j+\frac{1}{2}}}, \\ N & := \frac{1}{8}\frac{\left((\lambda a^n_{j+1}\sigma^n_{j+1})^2 f_{uu}(k_{j+1}, \zeta_3) - (\lambda a_j \sigma^n_j)^2 f_{uu}(k_j, \zeta_2)\right)}{\Delta u^n_{j+\frac{1}{2}}}. \end{aligned} \tag{5.5.34}$$

Next, in order to simplify L, we consider the following Taylor series expansions

$$\begin{aligned} f(k_j, u^n_j) = {} & f(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) - \frac{1}{2}\Delta k_{j+\frac{1}{2}} f_k(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) - \frac{1}{2}\Delta u^n_{j+\frac{1}{2}} f_u(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) \\ & + \frac{1}{8}(\Delta u^n_{j+\frac{1}{2}})^2 f_{uu}(c_6, \zeta_6) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \\ f(k_{j+1}, u^n_{j+1}) = {} & f(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) + \frac{1}{2}\Delta k_{j+\frac{1}{2}} f_k(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) + \frac{1}{2}\Delta u^n_{j+\frac{1}{2}} f_u(\tilde{k}_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) \\ & + \frac{1}{8}(\Delta u^n_{j+\frac{1}{2}})^2 f_{uu}(c_7, \zeta_7) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \end{aligned} \tag{5.5.35}$$

where $c_6 \in \mathcal{I}(k_j, \tilde{k}_{j+\frac{1}{2}})$, $c_7 \in \mathcal{I}(k_{j+1}, \tilde{k}_{j+\frac{1}{2}})$, $\zeta_6 \in \mathcal{I}(u_j^n, \tilde{u}_{j+\frac{1}{2}}^n)$, $\zeta_7 \in \mathcal{I}(u_{j+1}^n, \tilde{u}_{j+\frac{1}{2}}^n)$. Now, in view of(5.5.35), the term L in (5.5.33) reads as

$$L = \tilde{a}_{j+\frac{1}{2}}^n + \frac{1}{8}\Delta u_{j+\frac{1}{2}}^n \left(f_{uu}(c_7, \zeta_7) - f_{uu}(c_6, \zeta_6)\right) + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n}. \tag{5.5.36}$$

Further, the expressions in (5.5.28) allow us to write the term M as

$$\begin{aligned} M = & \frac{1}{\lambda}\left(\frac{1}{8} - \frac{1}{2}\beta^2\right)\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n} - \frac{\beta}{2}\left(\sigma_{j+1}^n f_{uu}(c_5, \zeta_5) + \sigma_j^n f_{uu}(c_4, \zeta_4)\right) \\ & - \frac{1}{8}\left(\lambda\left(\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)(\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_5, \zeta_5))^2 - \lambda\left(\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)(\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_4, \zeta_4))^2\right) \\ & + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n}. \end{aligned} \tag{5.5.37}$$

Combining the expressions (5.5.36) and (5.5.37), we may write (5.5.33) as

$$\lambda\frac{\Delta g_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n} = \beta + \frac{1}{8}\left(1 - 4\beta^2\right)\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n} + \mathcal{A}_4 + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n}, \tag{5.5.38}$$

where we define

$$\begin{aligned} \mathcal{A}_4 := & \frac{\lambda}{8}\Delta u_{j+\frac{1}{2}}^n \left(f_{uu}(c_7, \zeta_7) - f_{uu}(c_6, \zeta_6)\right) - \lambda\frac{\beta}{2}\left(\sigma_{j+1}^n f_{uu}(c_5, \zeta_5) + \sigma_j^n f_{uu}(c_4, \zeta_4)\right) \\ & - \frac{1}{8}\left[\left(\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)(\lambda\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_5, \zeta_5))^2 - \left(\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)(\lambda\Delta u_{j+\frac{1}{2}}^n f_{uu}(c_4, \zeta_4))^2\right] + \lambda N. \end{aligned} \tag{5.5.39}$$

In view of (5.5.38), the expression (5.5.32) now reads as

$$\begin{aligned} I = & -\frac{1}{8}(\Delta u_{j+\frac{1}{2}}^n)^2(1 - 4\beta^2)\left[1 - \frac{1}{16}(1 - 4\beta^2)\left(\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)^2 - \beta\left(\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right] \\ & - \frac{1}{8}(\Delta u_{j+\frac{1}{2}}^n)^2\left[-\mathcal{A}_4\left(8\beta + (1 - 4\beta^2)\left(\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right) - 4(\mathcal{A}_4)^2\right] + \mathcal{O}(\Delta k_{j+\frac{1}{2}}). \end{aligned} \tag{5.5.40}$$

Further, combining (5.5.31) and (5.5.40), we write

$$\begin{aligned} I + \bar{J} = & -\frac{1}{8}\eta_{j+\frac{1}{2}}^n(\Delta u_{j+\frac{1}{2}}^n)^2 + \frac{1}{16}\theta_{j+\frac{1}{2}}^n\lambda(\Delta u_{j+\frac{1}{2}}^n)^3 + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) \\ & + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}), \end{aligned} \tag{5.5.41}$$

where

$$\eta_{j+\frac{1}{2}}^n := (1 - 4\beta^2)\left[1 - \frac{1}{16}(1 - 4\beta^2)\left(\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}\right)^2 - (\beta + \mathcal{A}_4)\left(\frac{\Delta\sigma_{j+\frac{1}{2}}^n}{\Delta u_{j+\frac{1}{2}}^n}\right) - \frac{(\sigma_j^n + \sigma_{j+1}^n)}{2\Delta u_{j+\frac{1}{2}}^n}\right], \tag{5.5.42}$$

$$\theta^n_{j+\frac{1}{2}} := \mathcal{A}_2 - \mathcal{A}_3 + 16\frac{\beta\mathcal{A}_4}{\lambda\Delta u^n_{j+\frac{1}{2}}} + 8\frac{(\mathcal{A}_4)^2}{\lambda\Delta u^n_{j+\frac{1}{2}}}. \tag{5.5.43}$$

Now, our immediate aim is to reformulate the expression (5.5.41) in the form

$$I + \bar{J} = -\frac{1}{8}\tilde{\eta}^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 + \frac{1}{16}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3 + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) \tag{5.5.44}$$

$$+ \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}), \tag{5.5.45}$$

with the coefficients $\tilde{\eta}^n_{j+\frac{1}{2}}, \tilde{\theta}^n_{j+\frac{1}{2}} \geq 0$. To achieve this, we begin with decomposing the term $\mathcal{A}_3$ as

$$\mathcal{A}_3 = \tilde{\mathcal{A}}_3 + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \tag{5.5.46}$$

where we define

$$\begin{aligned}\tilde{\mathcal{A}}_3 := &\left[4\beta\frac{\sigma^n_{j+1}}{\Delta u^n_{j+\frac{1}{2}}}f_{uu}(c_5,\zeta_5) - 4\beta\frac{\sigma^n_j}{\Delta u^n_{j+\frac{1}{2}}}f_{uu}(c_4,\zeta_4)\right]\\ &+\left[\lambda(f_{uu}(c_4,\zeta_4))(2\tilde{a}^n_{j+\frac{1}{2}} - 2a^n_j)\left(\frac{\sigma^n_j}{\Delta u^n_{j+\frac{1}{2}}}\right)\right.\\ &\qquad\left.+\lambda f_{uu}(c_5,\zeta_5)(2a^n_{j+1} - 2\tilde{a}^n_{j+\frac{1}{2}})\left(\frac{\sigma^n_{j+1}}{\Delta u^n_{j+\frac{1}{2}}}\right)\right].\end{aligned}$$

Here, the term $\tilde{\mathcal{A}}_3$ can be bounded as

$$|\tilde{\mathcal{A}}_3| \leq 8\gamma_2(|\beta| + \kappa). \tag{5.5.47}$$

This is derived using (5.1.3) and the hypothesis **H2**, along with the fact that $0 \leq \frac{\sigma^n_{j+1}}{\Delta u^n_{j+\frac{1}{2}}} \leq 1$ (see (5.2.2)). Using the same arguments, we obtain a bound on the term $\mathcal{A}_4$ in (5.5.39) as

$$|\mathcal{A}_4| \leq C_0\lambda|\Delta u^n_{j+\frac{1}{2}}|\gamma_2, \tag{5.5.48}$$

where $C_0 := \dfrac{1}{4} + \kappa + \dfrac{\kappa^2}{4} + \dfrac{1}{2}\lambda C_{u_0}\gamma_2$. In addition to this bound, we also need to split $\mathcal{A}_4$ into a suitable form. To do this, we rewrite the term λN in (5.5.39) by using the expressions for $f_{uu}(k_j,\zeta_2)$ and $f_{uu}(k_{j+1},\zeta_3)$ taken from the Taylor expansions given in (5.5.21),

$$\lambda N = E_1 + E_2 + E_3, \tag{5.5.49}$$

where

$$\begin{aligned}E_1 &:= \lambda\frac{\left(f(k_{j+1},u^{n+\frac{1}{2}}_{j+1}) - f(k_j,u^{n+\frac{1}{2}}_j)\right)}{\Delta u^n_{j+\frac{1}{2}}},\\ E_2 &:= -\lambda\frac{\left(f(k_{j+1},u^n_{j+1}) - f(k_j,u^n_j)\right)}{\Delta u^n_{j+\frac{1}{2}}},\ \ E_3 := \lambda^2\frac{\left(a^2_{j+1}(\sigma^n_{j+1})^2 - a^2_j(\sigma^n_j)^2\right)}{2\Delta u^n_{j+\frac{1}{2}}}.\end{aligned} \tag{5.5.50}$$

Adding and subtracting $\lambda f(k_j, u_{j+1}^{n+\frac{1}{2}})$ to the numerator and applying the mean value theorem, we may write E_1 as

$$E_1 = \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n} + \lambda f_u(k_j, \zeta_8)\left(1 - \frac{\lambda}{2}\left(a_{j+1}^n \frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n} - a_j^n \frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right), \tag{5.5.51}$$

for some $c_8 \in \mathcal{I}(k_j, k_{j+1})$ and $\zeta_8 \in \mathcal{I}(u_j^{n+\frac{1}{2}}, u_{j+1}^{n+\frac{1}{2}})$, where we have used the definition of $u_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ from (5.2.5). Similarly, adding and subtracting $\lambda(f(k_j, u_{j+1}^n))$, we may write E_2 as

$$E_2 = \lambda\frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n} - \lambda f_u(k_j, \zeta_9), \tag{5.5.52}$$

for some $c_9 \in \mathcal{I}(k_j, k_{j+1})$ and $\zeta_9 \in \mathcal{I}(u_j^n, u_{j+1}^n)$.

Now, substituting the expressions for E_1, E_2 and E_3 from (5.5.51), (5.5.52) and (5.5.50), respectively, and using the relations (5.5.27) and (5.5.35) for the terms $f_{uu}(c_4, \zeta_4)$, $f_{uu}(c_5, \zeta_5)$, $f_{uu}(c_6, \zeta_6)$ and $f_{uu}(c_7, \zeta_7)$, the term $\mathcal{A}_4$ can be reformulated as

$$\mathcal{A}_4 = \tilde{\mathcal{A}}_4 + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u_{j+\frac{1}{2}}^n}, \tag{5.5.53}$$

where $\tilde{\mathcal{A}}_4 := \tilde{\mathcal{A}}_4^1 + \tilde{\mathcal{A}}_4^2 + \tilde{\mathcal{A}}_4^3 + \tilde{\mathcal{A}}_4^4 + \tilde{\mathcal{A}}_4^5 + \tilde{\mathcal{A}}_4^6$, and

$$\begin{aligned}
\tilde{\mathcal{A}}_4^1 &:= \frac{1}{2}\lambda\left[f_u(k_{j+1}, \zeta_{11}) + f_u(k_j, \zeta_{10}) - 2f_u(\tilde{k}_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^n))\right],\\
\tilde{\mathcal{A}}_4^2 &:= -\lambda\beta\left[\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}(a_{j+1}^n - \tilde{a}_{j+\frac{1}{2}}^n) + \frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}(\tilde{a}_{j+\frac{1}{2}}^n - a_j^n)\right],\\
\tilde{\mathcal{A}}_4^3 &:= -\frac{1}{8}\lambda^2\left[\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n}\left(4(a_{j+1}^n)^2 + 4(\tilde{a}_{j+\frac{1}{2}}^n)^2 - 8a_{j+1}^n\tilde{a}_{j+\frac{1}{2}}^n\right)\right]\\
&\quad + \frac{1}{8}\lambda^2\left[\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\left(4(a_j^n)^2 + 4(\tilde{a}_{j+\frac{1}{2}}^n)^2 - 8a_j^n\tilde{a}_{j+\frac{1}{2}}^n\right)\right],\\
\tilde{\mathcal{A}}_4^4 &:= \lambda f_u(k_j, \zeta_8)\left[1 - \frac{\lambda}{2}\left(a_{j+1}^n\frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n} - a_j^n\frac{\sigma_j^n}{\Delta u_{j+\frac{1}{2}}^n}\right)\right],\\
\tilde{\mathcal{A}}_4^5 &:= -\lambda f_u(k_j, \zeta_9), \quad \tilde{\mathcal{A}}_4^6 := \lambda^2\frac{\left(a_{j+1}^2(\sigma_{j+1}^n)^2 - a_j^2(\sigma_j^n)^2\right)}{2\Delta u_{j+\frac{1}{2}}^n}.
\end{aligned}$$

Using (5.1.3), (5.4.3) and the fact that $0 \leq \frac{\sigma_{j+1}^n}{\Delta u_{j+\frac{1}{2}}^n} \leq 1$, we easily obtain the bounds

$$\begin{aligned}
&|\tilde{\mathcal{A}}_4^1| \leq 2\kappa, \quad |\tilde{\mathcal{A}}_4^2| \leq 4\kappa^2, \quad |\tilde{\mathcal{A}}_4^3| \leq 4\kappa^2, \quad |\tilde{\mathcal{A}}_4^4| \leq \kappa(1+\kappa),\\
&|\tilde{\mathcal{A}}_4^5| \leq \kappa \quad \text{and} \quad |\tilde{\mathcal{A}}_4^6| \leq 2C_{u_0}\kappa^2,
\end{aligned}$$

and hence

$$|\tilde{\mathcal{A}}_4| \le C_1\kappa, \tag{5.5.54}$$

where $C_1 := 4 + (9 + 2C_{u_0})\kappa$. Upon rewriting (5.5.43) using the expressions (5.5.46) and (5.5.53) for $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively, we arrive at

$$\theta^n_{j+\frac{1}{2}} = \tilde{\theta}^n_{j+\frac{1}{2}} + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{(\Delta u^n_{j+\frac{1}{2}})^2}, \tag{5.5.55}$$

where we define

$$\tilde{\theta}^n_{j+\frac{1}{2}} := \mathcal{A}_2 - \tilde{\mathcal{A}}_3 + 16\frac{\beta\mathcal{A}_4}{\lambda\Delta u^n_{j+\frac{1}{2}}} + 8\frac{\mathcal{A}_4\tilde{\mathcal{A}}_4}{\lambda\Delta u^n_{j+\frac{1}{2}}}. \tag{5.5.56}$$

Analogously, in view of (5.5.53), (5.5.42) yields

$$\eta^n_{j+\frac{1}{2}} = \tilde{\eta}^n_{j+\frac{1}{2}} + \frac{\mathcal{O}(\Delta k_{j+\frac{1}{2}})}{\Delta u^n_{j+\frac{1}{2}}},$$

where we define

$$\begin{aligned}\tilde{\eta}^n_{j+\frac{1}{2}} := (1-4\beta^2)\Bigg[&1 - \frac{1}{16}(1-4\beta^2)\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2 \\ &-(\beta+\tilde{\mathcal{A}}_4)\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right) - \frac{(\sigma^n_j + \sigma^n_{j+1})}{2\Delta u^n_{j+\frac{1}{2}}}\Bigg],\end{aligned} \tag{5.5.57}$$

Hence, we obtain the desired reformulation (5.5.44) of (5.5.41).

Next, we show that $\tilde{\eta}^n_{j+\frac{1}{2}} \ge 0$ and $\tilde{\theta}^n_{j+\frac{1}{2}} > 0$, for all $j \in \mathbb{Z}$. First, we consider the term $\tilde{\eta}^n_{j+\frac{1}{2}}$, and note that $1 - 4\beta^2 \ge 0$, by the CFL condition (5.5.3). Now, using the bound (5.5.54), a portion of the term $\tilde{\eta}^n_{j+\frac{1}{2}}$ can be bounded as

$$\begin{aligned}1 &- \frac{1}{16}(1-4\beta^2)\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2 - (\beta+\tilde{\mathcal{A}}_4)\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right) - \frac{(\sigma^n_j+\sigma^n_{j+1})}{2\Delta u^n_{j+\frac{1}{2}}} \\ &\ge \left(\frac{1}{2} - \frac{1}{16}(1-4\beta^2)\Big|\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\Big| - |\beta+\tilde{\mathcal{A}}_4|\right)\left|\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right| + 1 \\ &\quad - \left(\left|\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{2\Delta u^n_{j+\frac{1}{2}}}\right| + \frac{(\sigma^n_j+\sigma^n_{j+1})}{2\Delta u^n_{j+\frac{1}{2}}}\right) \\ &\ge \left(\frac{1}{2} - \frac{1}{16}(1-4\beta^2) - |\beta + C_1\kappa|\right)\left|\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right| \ge 0.\end{aligned} \tag{5.5.58}$$

Here, we use the bound $(1 + C_1)\kappa \le \dfrac{7}{16}$, that follows from the CFL condition (5.5.3). Thus, we obtain $\tilde{\eta}^n_{j+\frac{1}{2}} \ge 0$, for all $j \in \mathbb{Z}$.

Now, collecting the estimates (5.5.25), (5.5.47), (5.5.48) and (5.5.54), we deduce that

$$\tilde{\theta}^n_{j+\frac{1}{2}} \geq \frac{4}{3}\gamma_1 - 8\gamma_2(|\beta| + \kappa) - 16|\beta|\gamma_2 C_0 - 8\gamma_2\kappa C_0 C_1 \geq \gamma_1 > 0, \text{ for all } j \in \mathbb{Z}, \tag{5.5.59}$$

as the CFL condition (5.5.3) implies that $\kappa \leq \dfrac{\gamma_1}{24\gamma_2\left(2(1+C_0) + C_0C_1\right)}$ for $\Delta t \leq \Delta x$. We also obtain an upper bound on $\tilde{\theta}^n_{j+\frac{1}{2}}$, by invoking (5.5.25), (5.5.47), (5.5.48) and (5.5.54), as follows

$$\tilde{\theta}^n_{j+\frac{1}{2}} \leq C_2, \text{ where } C_2 := 2\kappa^2 + \frac{4}{3}\gamma_2 + 8\kappa\gamma_2(2 + 2C_0 + C_0C_1). \tag{5.5.60}$$

Finally, in light of the equation (5.5.44), the expression (5.5.17) reduces to

$$\begin{aligned} &S(k_{j+\frac{1}{2}}, u^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(k_j, u^n_j) + S(k_{j+1}, u^n_{j+1})\right) + \lambda\left(G(k_{j+1}, u^n_{j+1}) - G(k_j, u^n_j)\right)\\ &= -\frac{1}{8}\tilde{\eta}^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 + \frac{1}{16}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3 + \Phi_{j+\frac{1}{2}}, \end{aligned} \tag{5.5.61}$$

where $\Phi_{j+\frac{1}{2}} = \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+1} - k_{j+\frac{1}{2}})$. Summing (5.5.61) over $n = 0, 1, \ldots, N-1$ and $j + \frac{n}{2} \in \mathbb{Z}$ with $|j| \leq J$, and with the choice $S(k,u) = \dfrac{u^2}{2}$ yields

$$\begin{aligned} &\frac{1}{2}\sum_{|j|\leq J}(u^N_j)^2 - \frac{1}{2}\sum_{|j|\leq J}(u^0_j)^2 + \lambda\sum_{n=0}^{N-1}\left(G(k_J, u^n_J) - G(k_{-J}, u^n_{-J})\right) \\ &= -\frac{1}{8}\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\tilde{\eta}^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 + \frac{1}{16}\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3_- \\ &\quad + \frac{1}{16}\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3_+ + \sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\Phi_{j+\frac{1}{2}}. \end{aligned} \tag{5.5.62}$$

Rearranging (5.5.62) and taking into account $\tilde{\eta}^n_{j+\frac{1}{2}} \geq 0$ (see (5.5.58)), it follows that

$$\begin{aligned} &-\frac{1}{16}\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3_- \\ &\leq \frac{1}{2}\sum_{|j|\leq J}(u^0_j)^2 - \frac{1}{2}\sum_{|j|\leq J}(u^N_j)^2 - \lambda\sum_{n=0}^{N-1}\left(G(k_J, u^n_J) - G(k_{-J}, u^n_{-J})\right) \\ &\quad + \frac{1}{16}\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\tilde{\theta}^n_{j+\frac{1}{2}}\lambda(\Delta u^n_{j+\frac{1}{2}})^3_+ + C_3 N\|k\|_{BV}. \end{aligned} \tag{5.5.63}$$

for some constant $C_3 \geq 0$.

Now, summing (5.5.2) of Lemma 5.5.1 over $n = 0, \ldots, N-1$, we see that

$$\frac{1}{500}\lambda\gamma_1\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}(\Delta u^n_{j-\frac{1}{2}})^3_+ \leq \frac{1}{500}\lambda\gamma_1\sum_{n=0}^{N-1}\sum_{j+\frac{n}{2}\in\mathbb{Z}}(\Delta u^n_{j-\frac{1}{2}})^3_+ \tag{5.5.64}$$

$$
\begin{aligned}
&\le \sum_{n=0}^{N-1} \sum_{j+\frac{n}{2}\in\mathbb{Z}} \left((\Delta u^n_{j-\frac{1}{2}})^2_+ - \left(\Delta u^{n+1}_j\right)^2_+ \right) + \sum_{n=0}^{N-1} \Psi \|k\|_{BV} \\
&\le \sum_{j\in\mathbb{Z}} (\Delta u^0_{j-\frac{1}{2}})^2_+ - \sum_{j\in\mathbb{Z}} \left(\Delta u^N_j\right)^2_+ + \sum_{n=0}^{N-1} \Psi \|k\|_{BV} \\
&\le \sum_{j\in\mathbb{Z}} (\Delta u^0_{j-\frac{1}{2}})^2_+ + \sum_{n=0}^{N-1} \Psi \|k\|_{BV}, \\
&\le 2\|u_0\| \|u_0\|_{BV} + \Psi \|k\|_{BV} N.
\end{aligned}
$$

The estimate (5.5.64) derived above, together with the bound $\gamma_1 < \tilde{\theta}^n_{j+\frac{1}{2}} \le C_2$ given in (5.5.59) and (5.5.60), applied to (5.5.63), yields

$$
\begin{aligned}
&- \frac{1}{16}\gamma_1 \lambda \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J \\ j+\frac{n}{2}\in\mathbb{Z}}} (\Delta u^n_{j+\frac{1}{2}})^3_- && (5.5.65) \\
&\le \frac{1}{2} \sum_{|j|\le J} (u^0_j)^2 - \frac{1}{2} \sum_{|j|\le J} (u^N_j)^2 - \lambda \sum_{n=0}^{N-1} \left(G(k_J, u^n_J) - G(k_{-J}, u^n_{-J}) \right) \\
&\quad + \frac{125}{4}\frac{C_2}{\gamma_1} \left(2\|u_0\|\|u_0\|_{BV}\right) + \left(\frac{125}{4}\frac{C_2}{\gamma_1}\Psi N + C_3 N \right) \|k\|_{BV} \\
&\le \frac{1}{2}\|u_0\|^2(2J+1) + 2\lambda \|G\| N + \frac{125}{4}\frac{C_2}{\gamma_1} \left(2\|u_0\|\|u_0\|_{BV}\right) && (5.5.66) \\
&\quad + \left(\frac{125}{4}\frac{C_2}{\gamma_1}\Psi N + C_3 N \right) \|k\|_{BV}.
\end{aligned}
$$

Observe that

$$
\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J \\ j+\frac{n}{2}\in\mathbb{Z}}} |\Delta u^n_{j+\frac{1}{2}}|^3 = \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J \\ j+\frac{n}{2}\in\mathbb{Z}}} \left((\Delta u^n_{j-\frac{1}{2}})^3_+ - (\Delta u^n_{j-\frac{1}{2}})^3_- \right). \tag{5.5.67}
$$

Now, using the estimates (5.5.64) and (5.5.65) in the above equation, we obtain the desired estimate (5.5.4) with

$$
\begin{aligned}
C(X,T) := &\frac{C_4}{\lambda\gamma_1} 1000 \left(1 + \frac{C_2}{\gamma_1}\right) \|u_0\|\|u_0\|_{BV} + \frac{32}{\lambda\gamma_1}\|u_0\|^2 X \\
&+ \frac{32}{\lambda\gamma_1}\|G\| T + \frac{1}{\lambda^2\gamma_1} T \left(500\Psi + 500\frac{C_2}{\gamma_1}\Psi + 16 C_3 \right) \|k\|_{BV},
\end{aligned}
$$

when $\Delta x \le C_4$, for a constant $C_4 > 0$. This concludes the proof. □

Now, we derive a quadratic estimate on the spatial differences of the approximate solutions.

Lemma 5.5.3. *(Quadratic estimate) Let the initial datum $u_0 \in (\mathrm{L}^\infty \cap \mathrm{BV})(\mathbb{R})$ be such that $\|u_0\| \le C_{u_0}$. Consider the cell-average approximate solutions $\{u_j^n\}$ generated by the scheme* (5.2.4) *and define*

$$\begin{aligned}\nu^n_{j+\frac{1}{2}} := \frac{1}{8}(1-4(\beta^n_{j+\frac{1}{2}})^2)\Bigg[1 &- \frac{1}{16}(1-4(\beta^n_{j+\frac{1}{2}})^2)\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2 \\ &- \beta^n_{j+\frac{1}{2}}\left(\frac{\Delta\sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right) - \frac{\sigma^n_{j+1}+\sigma^n_j}{2\Delta u^n_{j+\frac{1}{2}}}\Bigg] f_{uu}(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}),\end{aligned} \tag{5.5.68}$$

with $\beta^n_{j+\frac{1}{2}} := \lambda\tilde{a}^n_{j+\frac{1}{2}}$, where $\tilde{a}^n_{j+\frac{1}{2}}$ and $\tilde{u}^n_{j+\frac{1}{2}}$ are as in (5.5.26)*. If the CFL condition* (5.5.3) *holds, then we have:*
(i) $\nu^n_{j+\frac{1}{2}} \ge 0$ for $j \in \mathbb{Z}$, and
(ii) For any fixed T, X, N and J as in Lemma 5.5.2, the following estimate holds:

$$\Delta x \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}} \nu^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 \le K(X,T), \tag{5.5.69}$$

where $K(X,T)$ is a constant independent of Δx.

Proof. Recall the expression (5.5.17):

$$\bar{E}^n_{j+\frac{1}{2}} = I + \bar{J} + R^n_1 + R^n_2 + \tilde{R}^n_1 + \tilde{R}^n_2, \tag{5.5.70}$$

where I and $\bar{J}$ is as in (5.5.19) and (5.5.14), respectively. Choosing $S(k,u) = f(k,u) - f(k,c)$ in (5.5.19) and using the fact that $\bar{g}'_{j+\frac{1}{2}} = \frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}$, we write

$$I = -(\Delta u^n_{j+\frac{1}{2}})^2\left(\frac{1}{4} - \left(\lambda\frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2\right)\int_0^1\int_0^1 s f_{uu}(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}(r,s))\,\mathrm{d}s\,\mathrm{d}r.$$

Now, owing to hypothesis **H2**, we expand f_{uu} in the second variable about the point $(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}})$ using a Taylor series, and write the term I as

$$I = -\frac{1}{8}(\Delta u^n_{j+\frac{1}{2}})^2\left(1 - \left(2\lambda\frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)^2\right) f_{uu}(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) + \mathcal{O}((\Delta u_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \tag{5.5.71}$$

where $\tilde{u}^n_{j+\frac{1}{2}}$ is defined as in (5.5.26). The above expression uses the fact that

$$\begin{aligned}u_{j+\frac{1}{2}}(r,s) - \tilde{u}^n_{j+\frac{1}{2}} &= \frac{1}{2}\Delta u^n_{j+\frac{1}{2}}\,(1-s(1+r)) - \lambda\frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}s(1-r)\Delta u^n_{j+\frac{1}{2}} \\ &= \mathcal{O}(\Delta u^n_{j+\frac{1}{2}}) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}).\end{aligned}$$

Further, the choice $S(k,u) = f(k,u) - f(k,c)$ and the trapezoidal rule of integration along with (5.5.22) simplify $\bar{J}$ in (5.5.14) to

$$
\begin{aligned}
\bar{J} &= \lambda \int_0^1 f_{uu}(k_{j+\frac{1}{2}}, u(s)) \Big(f(k_{j+\frac{1}{2}}, u(s)) - \bar{g}_{j+\frac{1}{2}}(u(s)) \Big) u'(s)\,\mathrm{d}s \\
&= -\frac{\lambda}{2}\Delta u^n_{j+\frac{1}{2}} \Big[f_{uu}(k_{j+\frac{1}{2}}, u^n_{j+1}) \Big(f(k_{j+\frac{1}{2}}, u^n_{j+1}) - g(k_{j+1}, u^n_{j+1}) \Big) \\
&\quad + f_{uu}(k_{j+\frac{1}{2}}, u^n_j) \Big(f(k_{j+\frac{1}{2}}, u^n_j) - g(k_j, u^n_j) \Big) \Big] + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) \\
&= -\frac{\lambda}{2}\Delta u^n_{j+\frac{1}{2}} \Big[f_{uu}(k_{j+\frac{1}{2}}, u^n_{j+1}) \Big(\frac{\lambda}{2}(a^n_{j+1})^2 \sigma^n_{j+1} - \frac{1}{8}(\lambda a^n_{j+1}\sigma^n_{j+1})^2 f_{uu}(k_{j+1}, \zeta_3) - \frac{1}{8\lambda}\sigma^n_{j+1} \Big) \\
&\quad + f_{uu}(k_{j+\frac{1}{2}}, u^n_j) \Big(\frac{\lambda}{2}(a^n_j)^2 \sigma^n_j - \frac{1}{8}(\lambda a^n_j \sigma^n_j)^2 f_{uu}(k_j, \zeta_2) - \frac{1}{8\lambda}\sigma^n_j \Big) \Big] \\
&\quad + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}).
\end{aligned}
$$

Again, noting that $\sigma^n_j, \sigma^n_{j+1}$ are $\mathcal{O}(\Delta u^n_{j+\frac{1}{2}})$, replacing $(a^n_j)^2$ and $(a^n_{j+1})^2$ from their expression in (5.5.28), and subsequently expanding f_{uu} in a Taylor series about the point $(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}})$, the term $\bar{J}$ reduces to

$$
\begin{aligned}
\bar{J} &= \frac{1}{16}\Delta u^n_{j+\frac{1}{2}}(1 - 4(\beta^n_{j+\frac{1}{2}})^2) \Big[f_{uu}(k_{j+\frac{1}{2}}, u^n_{j+1})\sigma^n_{j+1} + f_{uu}(k_{j+\frac{1}{2}}, u^n_j)\sigma^n_j \Big] \\
&\quad + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}) \\
&= \frac{1}{16}(\Delta u^n_{j+\frac{1}{2}})^2 (1 - 4(\beta^n_{j+\frac{1}{2}})^2) f_{uu}(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) \frac{(\sigma^n_{j+1} + \sigma^n_j)}{\Delta u^n_{j+\frac{1}{2}}} + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) \\
&\quad + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}).
\end{aligned}
\tag{5.5.72}
$$

Now, we note that the term

$$
-\frac{1}{8}(\Delta u^n_{j+\frac{1}{2}})^2 \left(1 - \left(2\lambda \frac{\Delta g^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}} \right)^2 \right),
$$

which appears in (5.5.71), is nothing but the value of I obtained with the choice $S(k,u) = \frac{u^2}{2}$, see (5.5.32). Therefore, we replace this term in (5.5.71) with the right-hand side of (5.5.40). The resulting I of (5.5.71) is added with the term $\bar{J}$ in (5.5.72), to yield

$$
\begin{aligned}
I + \bar{J} = -\frac{1}{8}(1 - 4(\beta^n_{j+\frac{1}{2}})^2) &\Bigg[1 - \frac{1}{16}(1 - 4(\beta^n_{j+\frac{1}{2}})^2) \left(\frac{\Delta \sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}} \right)^2 \\
&- (\beta^n_{j+\frac{1}{2}} + \mathcal{A}_4) \left(\frac{\Delta \sigma^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}} \right) - \frac{(\sigma^n_j + \sigma^n_{j+1})}{2\Delta u^n_{j+\frac{1}{2}}} \Bigg] f_{uu}(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}})(\Delta u^n_{j+\frac{1}{2}})^2 \\
&+ \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}),
\end{aligned}
\tag{5.5.73}
$$

where $\mathcal{A}_4$ is as in (5.5.39). Since $\mathcal{A}_4 = \mathcal{O}(\Delta u^n_{j+\frac{1}{2}})$ (see (5.5.48)), the term above turns into

$$
I + \bar{J} = -\nu^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}), \tag{5.5.74}
$$

where $\nu^n_{j+\frac{1}{2}}$ is given by (5.5.68). Now, to show that $\nu^n_{j+\frac{1}{2}} \geq 0$, first we first observe that $1-4(\beta^n_{j+\frac{1}{2}})^2 \geq 0$, by the CFL condition (5.5.3) and $f_{uu}(k_{j+\frac{1}{2}}, \tilde{u}^n_{j+\frac{1}{2}}) \geq 0$, by hypothesis **H2**. This, together with an argument similar to (5.5.58) now implies that $\nu^n_{j+\frac{1}{2}} \geq 0$. Finally, observing that $R^n_1 + R^n_2 + \tilde{R}^n_1 + \tilde{R}^n_2 = \mathcal{O}(k_{j+\frac{1}{2}} - k_j) + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1})$ (see (5.5.7) and (5.5.16)) and calling the expression (5.5.74) in (5.5.17), we write

$$\begin{aligned}\bar{E}^n_{j+\frac{1}{2}} &= -\nu^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 + \mathcal{O}((\Delta u^n_{j+\frac{1}{2}})^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}}) + \mathcal{O}(k_{j+\frac{1}{2}} - k_j)\\ &\quad + \mathcal{O}(k_{j+\frac{1}{2}} - k_{j+1}).\end{aligned} \tag{5.5.75}$$

Recalling the notation (see (5.5.18)),

$$\bar{E}^n_{j+\frac{1}{2}} := S(k_{j+\frac{1}{2}}, u^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(k_j, u^n_j) + S(k_{j+1}, u^n_{j+1})\right) + \lambda\left(G(k_{j+1}, u^n_{j+1}) - G(k_j, u^n_j)\right),$$

summing (5.5.75) over $n = 0, 1, \ldots, N-1$, $j + \frac{n}{2} \in \mathbb{Z}$ with $|j| \leq J$, and multiplying by Δx, we obtain

$$\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} \nu^n_{j+\frac{1}{2}}(\Delta u^n_{j+\frac{1}{2}})^2 \tag{5.5.76}$$

$$\begin{aligned}&\leq \Delta x \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} \frac{1}{2}\left(S(k_j, u^0_j) + S(k_{j+1}, u^0_{j+1})\right) - \Delta x \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} S(k_{j+\frac{1}{2}}, u^N_{j+\frac{1}{2}})\\ &\quad + \Delta t \sum_{n=0}^{N-1} G(k_{-J}, u^n_{-J}) - \Delta t \sum_{n=0}^{N-1} G(k_J, u^n_J) + K_1 \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} |\Delta u^n_{j+\frac{1}{2}}|^3\\ &\quad + K_2 N \Delta x \|k\|_{BV},\end{aligned} \tag{5.5.77}$$

for some constants $K_1, K_2 \geq 0$. Finally, since $\|S\| \leq 2\|f\|$ for the choice $S(k,u) = f(k,u) - f(k,c)$, using the cubic estimate (5.5.4) from Lemma 5.5.2 and the boundedness of G in (5.5.76), we obtain the desired estimate (5.5.3), with

$$K(X,T) := 8X\|f\| + 2\|G\|T + K_1 C(X,T) + \frac{1}{\lambda} K_2 \|k\|_{BV} T,$$

and $C(X,T)$ is as in (5.5.4), thus completing the proof. □

5.6 Convergence of the second-order scheme

As a first step in proving convergence to a weak solution, we provide the $\mathrm{W}^{-1,2}_{\mathrm{loc}}$ compactness in the following lemma, which is necessary for applying the compensated compactness result in Theorem 5.3.1.

Lemma 5.6.1 ($\mathrm{W}^{-1,2}_{\mathrm{loc}}$compactness). *Let $u_0 \in (\mathrm{L}^\infty \cap \mathrm{BV})(\mathbb{R})$. Under the CFL condition* (5.5.3)*, for the approximate solutions u_Δ in* (5.2.8) *and the pairs (S_i, Q_i) considered in* (5.3.1)*, the sequence of distributions*

$$\{S_i(k(x), u_\Delta)_t + Q_i(k(x), u_\Delta)_x\}_{\Delta>0}\,,$$

is contained in a compact subset of $\mathrm{W}^{-1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$, for $i = 1, 2$.

Proof. Suppressing the index of (S_i, Q_i), we denote

$$\mathcal{L}^\Delta := -(S(u^\Delta)_t + Q(k(x), u^\Delta)_x)$$

which is defined by the action

$$\left\langle \mathcal{L}^\Delta, \phi\right\rangle := \int_{\mathbb{R}_+}\int_{\mathbb{R}} (S(k(x), u_\Delta)\phi_t + Q(k(x), u_\Delta)\phi_x)\,\mathrm{d}x\,\mathrm{d}t, \tag{5.6.1}$$

for $\phi \in W^{1,2}_{0,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$. Using k_Δ as defined in (5.2.8), we add and subtract $S(k_\Delta(x), u_\Delta)$ and $Q(k_\Delta(x), u_\Delta)$ in the integrand of (5.6.1), which in turn allows us to write $\mathcal{L}^\Delta = \hat{\mathcal{L}}^\Delta + \tilde{\mathcal{L}}^\Delta$, where

$$\begin{aligned}
\langle \hat{\mathcal{L}}^\Delta, \phi\rangle &:= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(S(k(x), u_\Delta) - S(k_\Delta(x), u_\Delta)\right)\phi_t \,\mathrm{d}x\ \mathrm{d}t\\
&\quad+ \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(Q(k(x), u_\Delta) - Q(k_\Delta(x), u_\Delta)\right)\phi_x \,\mathrm{d}x\ \mathrm{d}t,\\
\langle \tilde{\mathcal{L}}^\Delta, \phi\rangle &:= \int_{\mathbb{R}^+}\int_{\mathbb{R}} (S(k_\Delta(x), u_\Delta)\phi_t + Q(k_\Delta(x), u_\Delta)\phi_x)\,\mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

Now, we consider a bounded open subset Ω of $\mathbb{R}\times\mathbb{R}_+$ and let $X > 0$, $T > 0$ be such that $\Omega \subseteq [-X, X]\times[0, T]$. Further, choose smallest integers $J, N \in \mathbb{N}$ such that $J\Delta x > X + \Delta x$, and $N\Delta t > T$. Additionally, define $S^n_j := S(k_\Delta(x_j, t^n), u_\Delta(x_j, t^n)) = S(k^n_j, u^n_j)$, $Q^n_j := Q(k_\Delta(x_j, t^n), u_\Delta(x_j, t^n)) = Q(k^n_j, u^n_j)$.

Let $q \in (1, 2)$ and $p = \dfrac{q}{q-1}$. Now, applying Hölder's inequality to the term $\langle\hat{\mathcal{L}}^\Delta, \phi\rangle$ with $\phi \in W^{1,p}_0(\Omega)$, we obtain

$$\lim_{\Delta\to 0}|\langle\hat{\mathcal{L}}^\Delta, \phi\rangle| \leq \lim_{\Delta\to 0}\left(\|\partial_k S\| + \|\partial_k Q\|\right)\|k - k_\Delta\|_{L^q(\Omega)}\|\phi\|_{W^{1,p}_0(\Omega)} = 0,$$

from which we can conclude that

$$\{\hat{\mathcal{L}}^\Delta\}\text{ is compact in }\mathrm{W}^{-1,q}(\Omega)\text{ for any }q \in (1, 2). \tag{5.6.2}$$

Next, we focus on showing that $\{\tilde{\mathcal{L}}^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,q}(\Omega)$ for some $q \in (1, 2)$. We begin with applying summation by parts to expand the term $\left\langle \tilde{\mathcal{L}}^\Delta, \phi\right\rangle$ as

$$\left\langle \tilde{\mathcal{L}}^\Delta, \phi\right\rangle = \sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\int_{t^n}^{t^{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (S(k_\Delta(x), u_\Delta)\phi_t + Q(k_\Delta(x), u_\Delta)\phi_x)\,\mathrm{d}x\ \mathrm{d}t.$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \Bigg[\int_{x_{j-\frac12}}^{x_{j+\frac12}} S_j^n \left(\phi(x,t_{n+1}) - \phi(x,t_n)\right) \mathrm{d}x \\
&\quad + \int_{t_n}^{t_{n+1}} Q_j^n \left(\phi(x_{j+\frac12},t) - \phi(x_{j-\frac12},t)\right) \mathrm{d}t \Bigg] \\
&= \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_{j-\frac12}}^{x_{j+\frac12}} S_j^{N-1} \phi(x,N\Delta t)\,\mathrm{d}x - \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_{j-\frac12}}^{x_{j+\frac12}} S_j^{0} \phi(x,0)\,\mathrm{d}x \\
&\quad - \sum_{n=1}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \left[\int_{x_{j-\frac12}}^{x_j} \left(S_j^n - S_{j-\frac12}^{n-1}\right) \phi(x,t_n)\,\mathrm{d}x + \int_{x_j}^{x_{j+\frac12}} \left(S_j^n - S_{j+\frac12}^{n-1}\right) \phi(x,t_n)\,\mathrm{d}x \right] \\
&\quad - \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{t^n}^{t_{n+1}} \left(Q_{j+1}^n - Q_j^n\right) \phi(x_{j+\frac12},t)\,\mathrm{d}t.
\end{aligned}
$$

Further, denoting $\phi_j^n := \phi(x_j,t^n)$ and adding and subtracting suitable terms, we write

$$
\left\langle \tilde{\mathcal{L}}^\Delta, \phi \right\rangle =: < \tilde{\mathcal{L}}_0^\Delta, \phi > + < \tilde{\mathcal{L}}_1^\Delta, \phi > + < \tilde{\mathcal{L}}_2^\Delta, \phi > + < \tilde{\mathcal{L}}_3^\Delta, \phi > + < \tilde{\mathcal{L}}_4^\Delta, \phi >, \tag{5.6.3}
$$

where

$$
\begin{aligned}
< \tilde{\mathcal{L}}_0^\Delta, \phi > &:= \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_{j-\frac12}}^{x_{j+\frac12}} S_j^{N-1} \phi(x,N\Delta t)\,\mathrm{d}x - \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_{j-\frac12}}^{x_{j+\frac12}} S_j^{0} \phi(x,0)\,\mathrm{d}x, \qquad (5.6.4) \\
< \tilde{\mathcal{L}}_1^\Delta, \phi > &:= \sum_{n=1}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_{j-\frac12}}^{x_j} \left(S_j^n - S_{j-\frac12}^{n-1}\right) \left(\phi_j^n - \phi(x,t_n)\right) \mathrm{d}x, \\
< \tilde{\mathcal{L}}_2^\Delta, \phi > &:= \sum_{n=1}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \int_{x_j}^{x_{j+\frac12}} \left(S_j^n - S_{j+\frac12}^{n-1}\right) \left(\phi_j^n - \phi(x,t_n)\right) \mathrm{d}x, \\
< \tilde{\mathcal{L}}_3^\Delta, \phi > &:= \sum_{n=1}^{N} \sum_{\substack{|j|\le J\\ j+\frac{n-1}{2}\in\mathbb{Z}}} \int_{t_{n-1}}^{t_n} \left(Q_{j+\frac12}^{n-1} - Q_{j-\frac12}^{n-1}\right) \left(\phi_j^n - \phi(x_j,t)\right) \mathrm{d}t, \\
< \tilde{\mathcal{L}}_4^\Delta, \phi > &:= -\Delta x \sum_{n=1}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \phi_j^n \left(S_j^n - \frac12 \left(S_{j+\frac12}^{n-1} + S_{j-\frac12}^{n-1}\right) + \lambda \left(Q_{j+\frac12}^{n-1} - Q_{j-\frac12}^{n-1}\right) \right).
\end{aligned}
$$

Hereafter, we set $S(k,u) = S_2(k,u) = f(k,u) - f(k,c)$, $Q(k,u) = Q_2(k,u) = \int_c^u (f_u(k,\xi))^2\,\mathrm{d}\xi$. The proof for the case $S_1(u) = u - c$, $Q_1(k,u) = f(k,u) - f(k,c)$ follows analogously and is omitted. Now, using a change of index, we write the term

$< \tilde{\mathcal{L}}_4^\Delta, \phi >$ in (5.6.3) as

$$< \tilde{\mathcal{L}}_4^\Delta, \phi > = \Delta x \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \phi_{j+\frac{1}{2}}^{n+1} \left[S_{j+\frac{1}{2}}^{n+1} - \frac{1}{2}\left(S_{j+1}^n + S_j^n\right) + \lambda\left(Q_{j+1}^n - Q_j^n\right)\right]. \tag{5.6.5}$$

Proceeding as in the proof of Lemma 5.5.2 (see (5.5.5)-(5.5.19)) and using the same notations, we can write

$$\begin{aligned} & S_{j+\frac{1}{2}}^{n+1} - \frac{1}{2}\left(S_{j+1}^n + S_j^n\right) + \lambda\left(Q_{j+1}^n - Q_j^n\right) \\ &= I + \bar{J} - \lambda\Big(S_u(k_j, u_j^n)(f(k_j, u_j^n) - g(k_j, u_j^n)) \\ &\quad - S_u(k_{j+1}, u_{j+1}^n)(f(k_{j+1}, u_{j+1}^n) - g(k_{j+1}, u_{j+1}^n))\Big) + \tilde{R}_1 + \tilde{R}_2 + R_1 + R_2. \end{aligned} \tag{5.6.6}$$

Now, invoking (5.6.6) in (5.6.5), we write

$$< \tilde{\mathcal{L}}_4^\Delta, \phi > =< \tilde{\mathcal{L}}_{4,1}^\Delta, \phi > + < \tilde{\mathcal{L}}_{4,2}^\Delta, \phi > + < \tilde{\mathcal{L}}_{4,3}^\Delta, \phi >, \tag{5.6.7}$$

where we define

$$\begin{aligned} < \tilde{\mathcal{L}}_{4,1}^\Delta, \phi > &:= \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \Delta x \phi_{j+\frac{1}{2}}^{n+1}\left(I + \bar{J}\right), \\ < \tilde{\mathcal{L}}_{4,2}^\Delta, \phi > &:= -\sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \Delta x \phi_{j+\frac{1}{2}}^{n+1} \lambda\Big(S_u(k_j, u_j^n)(f(k_j, u_j^n) - g(k_j, u_j^n)) \\ &\qquad - S_u(k_{j+1}, u_{j+1}^n)(f(k_{j+1}, u_{j+1}^n) - g(k_{j+1}, u_{j+1}^n))\Big), \\ < \tilde{\mathcal{L}}_{4,3}^\Delta, \phi > &:= \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \Delta x \phi_{j+\frac{1}{2}}^{n+1}\left(\tilde{R}_1 + \tilde{R}_2 + R_1 + R_2\right). \end{aligned} \tag{5.6.8}$$

At this stage, by summarizing (5.6.3) and (5.6.8), we write

$$\begin{aligned} \left\langle \tilde{\mathcal{L}}^\Delta, \phi \right\rangle =&:< \tilde{\mathcal{L}}_0^\Delta, \phi > + < \tilde{\mathcal{L}}_1^\Delta, \phi > + < \tilde{\mathcal{L}}_2^\Delta, \phi > + < \tilde{\mathcal{L}}_3^\Delta, \phi > \\ & + < \tilde{\mathcal{L}}_{4,1}^\Delta, \phi > + < \tilde{\mathcal{L}}_{4,2}^\Delta, \phi > + < \tilde{\mathcal{L}}_{4,3}^\Delta, \phi >, \end{aligned} \tag{5.6.9}$$

and proceed with the proof of compactness of $\tilde{\mathcal{L}}^\Delta$ in two steps.

Step 1 (Compactness of $\tilde{\mathcal{L}}_{4,1}^\Delta, \tilde{\mathcal{L}}_{4,3}^\Delta$ and $\tilde{\mathcal{L}}_0^\Delta$): Using (5.5.74) from Lemma 5.5.3, we write

$$< \tilde{\mathcal{L}}_{4,1}^\Delta, \phi >= \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \Delta x \phi_{j+\frac{1}{2}}^{n+1}\left(-\nu_{j+\frac{1}{2}}^n (\Delta u_{j+\frac{1}{2}}^n)^2 + \mathcal{O}((\Delta u_{j+\frac{1}{2}}^n)^3) + \mathcal{O}(\Delta k_{j+\frac{1}{2}})\right). \tag{5.6.10}$$

Again, the cubic estimate (5.5.4) from Lemma 5.5.2 and the quadratic estimate (5.5.69) from Lemma 5.5.3, together with the hypothesis **H1** yield

$$
\begin{aligned}
|< \tilde{\mathcal{L}}_{4,1}^{\Delta}, \phi >| \leq \Delta x \|\phi\| \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} \nu_{j+\frac{1}{2}}^{n} (\Delta u_{j+\frac{1}{2}}^{n})^2 \\
+ \mathcal{K}_1 \Delta x \|\phi\| \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} |\Delta u_{j+\frac{1}{2}}^{n}|^3 + \mathcal{K}_2 \Delta x \|\phi\| \sum_{n=0}^{N-2} \sum_{\substack{|j|\leq J \\ j+\frac{n+1}{2}\in\mathbb{Z}}} |\Delta k_{j+\frac{1}{2}}| \\
\leq \mathcal{K}_3 \|\phi\|, \quad \text{for} \quad \phi \in C_0(\Omega),
\end{aligned}
\tag{5.6.11}
$$

and some constants $\mathcal{K}_1, \mathcal{K}_2 > 0$ and $K_3 := K(X,T) + \mathcal{K}_1 C(X,T) + \frac{1}{\lambda}\mathcal{K}_2 \|k\|_{BV} T$. Here, $C_0(\Omega)$ denotes the space of continuous functions on Ω that vanish at the boundary.

In the next step, we estimate $< \tilde{\mathcal{L}}_{4,3}^{\Delta}, \phi >$ for $\phi \in C_0(\Omega)$ by recalling (5.5.7), (5.5.16) and using hypothesis **H1**, as

$$
|< \tilde{\mathcal{L}}_{4,3}^{\Delta}, \phi >| \leq \|\phi\| \mathcal{K}_4 \sum_{n=0}^{N-2} \Delta x \|k\|_{\mathrm{BV}} \leq \mathcal{K}_5 \|\phi\|, \tag{5.6.12}
$$

where $\mathcal{K}_5 := \|k\|_{BV} \frac{\mathcal{K}_4}{\lambda} T$ and $\mathcal{K}_4 > 0$ is some constant. Combining the bounds (5.6.11) and (5.6.12), we obtain the following estimate

$$
\|\tilde{\mathcal{L}}_{4,1}^{\Delta}\|_{\mathcal{M}(\Omega)}, \|\tilde{\mathcal{L}}_{4,3}^{\Delta}\|_{\mathcal{M}(\Omega)} \leq \mathcal{K}_6 = \max\{\mathcal{K}_3, \mathcal{K}_5\}, \tag{5.6.13}
$$

as $\mathcal{M}(\Omega)$, the space of all bounded Radon measures on Ω, is the dual of the space $(C_0(\Omega), \|\cdot\|_\infty)$ (see [132] for more details). Similarly, dealing with the term $< \tilde{\mathcal{L}}_{0}^{\Delta}, \phi >$, we obtain a bound

$$
\|\tilde{\mathcal{L}}_{0}^{\Delta}\|_{\mathcal{M}(\Omega)} \leq \mathcal{K}_7, \tag{5.6.14}
$$

as $|< \tilde{\mathcal{L}}_{0}^{\Delta}, \phi >| \leq \mathcal{K}_7 \|\phi\|$ for some constant $\mathcal{K}_7 > 0$. By Sobolev's imbedding theorem (see Lemma 2.55, page 38 in [132]), we have the inclusion $\mathcal{M}(\Omega) \subset \mathrm{W}^{-1,q}(\Omega)$ with compact imbedding for $q \in [1,2)$. Consequently, from (5.6.13) and (5.6.14), it follows that the sets

$$
\{\tilde{\mathcal{L}}_{4,1}^{\Delta}\}_{\Delta>0}, \{\tilde{\mathcal{L}}_{4,3}^{\Delta}\}_{\Delta>0} \text{ and } \{\tilde{\mathcal{L}}_{0}^{\Delta}\}_{\Delta>0} \text{ are compact in } \mathrm{W}^{-1,q}(\Omega),\ \forall\, q \in [1,2). \tag{5.6.15}
$$

Step 2 (Compactness of $\tilde{\mathcal{L}}_{1}^{\Delta}, \tilde{\mathcal{L}}_{2}^{\Delta}, \tilde{\mathcal{L}}_{3}^{\Delta}$ and $\tilde{\mathcal{L}}_{4,2}^{\Delta}$): We first derive some estimates useful in bounding the term $< \tilde{\mathcal{L}}_{1}^{\Delta}, \phi >$ in (5.6.3). Observing that

$$
\begin{aligned}
\left| u_{j+\frac{1}{2}}^{n-\frac{1}{2}} - u_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right| &= \left| \Delta u_j^{n-1} - \frac{\lambda}{2} f_u(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n-1}) \sigma_{j+\frac{1}{2}}^{n-1} + \frac{\lambda}{2} f_u(k_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}^{n-1}) \sigma_{j-\frac{1}{2}}^{n-1} \right| \\
&\leq |\Delta u_j^{n-1}| \left(1 + \kappa\right),
\end{aligned}
\tag{5.6.16}
$$

we derive the following bound

$$
\begin{aligned}
&|u_j^n - u_{j-\frac{1}{2}}^{n-1}| \\
&\leq \left| \frac{1}{2}(u_{j+\frac{1}{2}}^{n-1} - u_{j-\frac{1}{2}}^{n-1}) - \frac{1}{8}(\sigma_{j+\frac{1}{2}}^{n-1} - \sigma_{j-\frac{1}{2}}^{n-1}) - \lambda\left(f(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n-1/2}) - f(k_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}^{n-1/2}) \right) \right| \\
&\leq \frac{1}{2}\left| u_{j+\frac{1}{2}}^{n-1} - u_{j-\frac{1}{2}}^{n-1} \right| + \frac{1}{8}\left(|\sigma_{j+\frac{1}{2}}^{n-1}| + |\sigma_{j-\frac{1}{2}}^{n-1}| \right) \\
&\quad + \left| \lambda\left(f(k_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^{n-1/2}) - f(k_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}^{n-1/2}) \right) \right| \\
&\leq \left(\frac{1}{2} + \frac{1}{4} \right) \left| \Delta u_j^{n-1} \right| + \lambda \|f_k\| |\Delta k_j| + \kappa \left| u_{j+\frac{1}{2}}^{n-\frac{1}{2}} - u_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right| \\
&\leq |\Delta u_j^{n-1}| \left(\frac{3}{4} + \kappa(1+\kappa) \right) + \lambda \|f_k\| |\Delta k_j|.
\end{aligned}
$$

Again, using the above estimate, we obtain

$$
\begin{aligned}
|S_j^n - S_{j-\frac{1}{2}}^{n-1}| &\leq \|S_k\| |k_j - k_{j-\frac{1}{2}}| + \|S_u\| |u_j^n - u_{j-\frac{1}{2}}^{n-1}| \\
&\leq \|S_k\| |k_j - k_{j-\frac{1}{2}}| + \mathcal{K}_8 \|S_u\| |\Delta u_j^{n-1}| + \|S_u\| \lambda \|f_k\| |\Delta k_j|,
\end{aligned}
$$

where $\mathcal{K}_8 := \dfrac{3}{4} + \kappa(1+\kappa)$. Using this estimate in $< \tilde{\mathcal{L}}_1^{\Delta}, \phi >$ and subsequently applying the Hölder's inequality, we write

$$
\begin{aligned}
|< \tilde{\mathcal{L}}_1^{\Delta}, \phi >| &\leq \sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} \mathcal{K}_8 \|S_u\| |\Delta u_j^{n-1}| \int_{x_{j-\frac{1}{2}}}^{x_j} |\phi_j^n - \phi(x, t_n)| \,\mathrm{d}x \\
&\quad + \sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} \Big(\|S_k\| |k_j - k_{j-\frac{1}{2}}| + \|S_u\| \lambda \|f_k\| |\Delta k_j| \Big) \int_{x_{j-\frac{1}{2}}}^{x_j} |\phi_j^n - \phi(x, t_n)| \,\mathrm{d}x \\
&\leq \frac{1}{2} \mathcal{K}_8 \sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} \|S_u\| |\Delta u_j^{n-1}| \|\phi\|_{C_0^\alpha} \Delta x^{\alpha+1} \\
&\quad + \frac{1}{2} \sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} \Big(\|S_k\| |k_j - k_{j-\frac{1}{2}}| + \|S_u\| \lambda \|f_k\| |\Delta k_j| \Big) \|\phi\|_{C_0^\alpha} \Delta x^{\alpha+1} \\
&\leq \frac{1}{2} \mathcal{K}_8 \|S_u\| \|\phi\|_{C_0^\alpha} \Delta x^{\alpha+\frac{2}{3}} \left(\Delta x \sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} |\Delta u_j^{n-1}|^3 \right)^{\frac{1}{3}} \left(\sum_{n=1}^{N-1} \sum_{\substack{|j| \leq J \\ j+\frac{n}{2} \in \mathbb{Z}}} 1 \right)^{\frac{2}{3}} \\
&\quad + \frac{1}{2\lambda} \left(\|S_k\| T + \|S_u\| \lambda \|f_k\| T \right) \|k\|_{BV} \|\phi\|_{C_0^\alpha} \Delta x^{\alpha},
\end{aligned}
$$

for $\phi \in C_0^\alpha(\Omega)$, which denotes the space of Hölder continuous functions. Finally, invoking Lemma 5.5.2 in the previous inequality, we obtain

$$
|< \tilde{\mathcal{L}}_1^{\Delta}, \phi >| \leq \frac{1}{2} C(X,T)^{\frac{1}{3}} \mathcal{K}_8 \|S_u\| \frac{(4XT)^{\frac{2}{3}}}{\lambda^{\frac{2}{3}}} \|\phi\|_{C_0^\alpha} \Delta x^{\alpha-\frac{2}{3}} \tag{5.6.17}
$$

$$+\frac{1}{2\lambda}\left(\|S_k\|T+\|S_u\|\lambda\|f_k\|T\right)\|k\|_{BV}\|\phi\|_{C_0^\alpha}\Delta x^\alpha \quad \text{for } \phi\in C_0^\alpha(\Omega).$$

Now, by the Sobolev's imbedding theorem (see [2]) we have the inclusion $W_0^{1,p}(\Omega)\hookrightarrow C_0^\alpha(\Omega)$ for $\alpha\in\left(0,1-\frac{2}{p}\right]$. Consequently, for a fixed $\alpha_1\in(2/3,1)$ and $p_1=\frac{2}{1-\alpha_1}$, there exists a constant $\mathcal{K}_{p_1}>0$, such that

$$\|\phi\|_{C_0^{\alpha_1}(\Omega)}\le \mathcal{K}_{p_1}\|\phi\|_{W_0^{1,p_1}(\Omega)}\ \forall\ \phi\in W_0^{1,p_1}(\Omega). \tag{5.6.18}$$

Using (5.6.18) in (5.6.17) amounts to

$$\begin{aligned}|<\tilde{\mathcal{L}}_1^\Delta,\phi>|\le{}&\frac{1}{2}C(X,T)^{\frac{1}{3}}\mathcal{K}_{p_1}\mathcal{K}_8\|S_u\|\frac{(4XT)^{\frac{2}{3}}}{\lambda^{\frac{2}{3}}}\|\phi\|_{W_0^{1,p_1}(\Omega)}\Delta x^{\alpha_1-\frac{2}{3}}\\&+\frac{1}{2\lambda}\mathcal{K}_{p_1}\left(\|S_k\|T+\|S_u\|\lambda\|f_k\|T\right)\|k\|_{BV}\|\phi\|_{W_0^{1,p_1}(\Omega)}\Delta x^{\alpha_1}.\end{aligned} \tag{5.6.19}$$

This implies that

$$\lim_{\Delta\to0}\|\tilde{\mathcal{L}}_1^\Delta\|_{\mathrm{W}^{-1,q_1}(\Omega)}\le\lim_{\Delta\to0}(\mathcal{K}_9\Delta x^{\alpha_1-\frac{2}{3}}+\mathcal{K}_{10}\Delta x^{\alpha_1})=0 \quad \text{for} \quad q_1=\frac{2}{1+\alpha_1}. \tag{5.6.20}$$

where

$$\mathcal{K}_9:=\frac{1}{2}C^{\frac{1}{3}}\mathcal{K}_{p_1}\mathcal{K}_8\|S_u\|\frac{(4XT)^{\frac{2}{3}}}{\lambda^{\frac{2}{3}}},\ \mathcal{K}_{10}:=\frac{1}{2\lambda}\mathcal{K}_{p_1}\left(\|S_k\|T+\|S_u\|\lambda\|f_k\|T\right)\|k\|_{BV}.$$

From this, we conclude that $\{\tilde{\mathcal{L}}_1^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,q_1}(\Omega)$, for $q_1=\dfrac{2}{1+\alpha_1}\in(1,2)$. In an entirely analogous way, we obtain that the set $\{\tilde{\mathcal{L}}_2^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,q_1}(\Omega)$.

Next, we consider the term $<\tilde{\mathcal{L}}_{4,2}^\Delta,\phi>$ from (5.6.8), and apply summation by parts to write

$$<\tilde{\mathcal{L}}_{4,2}^\Delta,\phi>=\lambda\Delta x\sum_{n=0}^{N-2}\sum_{\substack{|j|\le J\\ j+\frac{n+1}{2}\in\mathbb{Z}}}\left(\phi_{j-\frac{1}{2}}^{n+1}-\phi_{j+\frac{1}{2}}^{n+1}\right)S_u(k_j,u_j^n)(f(k_j,u_j^n)-g(k_j,u_j^n)). \tag{5.6.21}$$

From the expression (5.5.21) in the proof of Lemma 5.5.2, we deduce that

$$\lambda|f(k_j,u_j^n)-g(k_j,u_j^n)|=\lambda\left|\frac{\lambda}{2}(a_j^n)^2\sigma_j^n-\frac{1}{8}(\lambda a_j^n\sigma_j^n)^2f_{uu}(k_j,\zeta_2)-\frac{1}{8\lambda}\sigma_j^n\right|\le\mathcal{K}_{11}|\Delta u_{j+\frac{1}{2}}^n|, \tag{5.6.22}$$

where $\mathcal{K}_{11}:=\dfrac{\kappa^2}{2}+\dfrac{1}{4}C_{u_0}\kappa^2\gamma_2\lambda+\dfrac{1}{8}$. Inserting the estimate (5.6.22) into (5.6.21) and observing that $\lambda\|S_u\|\le\lambda\|f_u\|\le\kappa$, we apply Hölder's inequality on (5.6.21) along with the cubic estimate (5.5.2) of Lemma 5.5.2, to obtain

$$|\langle\tilde{\mathcal{L}}_{4,2}^\Delta,\phi\rangle|\le\kappa\mathcal{K}_{11}\left(\Delta x\sum_{n=0}^{N-2}\sum_{\substack{|j|\le J\\ j+\frac{n+1}{2}\in\mathbb{Z}}}|\Delta\phi_j^n|^{3/2}\right)^{2/3}\left(\Delta x\sum_{n=0}^{N-2}\sum_{\substack{|j|\le J\\ j+\frac{n+1}{2}\in\mathbb{Z}}}|\Delta u_{j+\frac{1}{2}}^n|^3\right)^{1/3} \tag{5.6.23}$$

$$\leq \kappa\mathcal{K}_{11}C(X,T)^{\frac{1}{3}}\left(\Delta x^{(1+\frac{3}{2}\alpha)}\|\phi\|_{C_0^\alpha}^{\frac{3}{2}\alpha}\sum_{n=0}^{N-2}\sum_{\substack{|j|\leq J\\ j+\frac{n+1}{2}\in\mathbb{Z}}}1\right)^{2/3}$$

$$\leq \kappa\mathcal{K}_{11}C(X,T)^{\frac{1}{3}}||\phi||_{C_0^\alpha}(\Delta x)^{\alpha+\frac{2}{3}}\frac{(4XT)^{\frac{2}{3}}}{\lambda^{\frac{2}{3}}(\Delta x)^{\frac{4}{3}}}\leq \mathcal{K}_{12}||\phi||_{C_0^\alpha}(\Delta x)^{\alpha-\frac{2}{3}}.$$

for $\phi\in C_0^\alpha(\Omega)$, where $\mathcal{K}_{12} := \kappa\mathcal{K}_{11}C(X,T)^{\frac{1}{3}}\left(\frac{4XT}{\lambda}\right)^{\frac{2}{3}}$ with $C(X,T)$ as in Lemma 5.5.2. To get the penultimate inequality in the above estimate, we have used the facts that $X+\Delta x\leq J\Delta x\leq X+2\Delta x$ and $(N-1)\Delta x\leq T$. Now, arguments similar to those in (5.6.17), (5.6.19) and (5.6.20), give compactness of $\{\tilde{\mathcal{L}}_{4,2}^\Delta\}_{\Delta>0}$ in $\mathrm{W}^{-1,q_1}(\Omega)$, for the same $q_1\in(1,2)$ as in the case of $\tilde{\mathcal{L}}_1^\Delta$ and $\tilde{\mathcal{L}}_2^\Delta$.

Next, we consider $\{\tilde{\mathcal{L}}_3^\Delta\}$, and obtain an estimate on the term $<\tilde{\mathcal{L}}_3^\Delta,\phi>$ using the observation

$$|Q_{j+\frac{1}{2}}^{n-1}-Q_{j-\frac{1}{2}}^{n-1}|\leq\|Q_k\||k_{j+\frac{1}{2}}-k_{j-\frac{1}{2}}|+\|Q_u\||\Delta u_j^n|,$$

as follows

$$\begin{aligned}|<\tilde{\mathcal{L}}_3^\Delta,\phi>|&\leq\lambda\|\phi\|_{C_0^\alpha}(\Delta x)^{\alpha+1}\|Q_k\|\|k\|_{BV}\left(\frac{2T}{\lambda\Delta x}\right)\\&\quad+\lambda\|\phi\|_{C_0^\alpha}(\Delta x)^{\alpha+\frac{2}{3}}\|Q_u\|\left(\Delta x\sum_{n=1}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}|\Delta u_j^{n-1}|^3\right)^{\frac{1}{3}}\left(\sum_{n=1}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}1\right)^{\frac{2}{3}}\\&\leq 2\|\phi\|_{C_0^\alpha}(\Delta x)^\alpha\|Q_k\|\|k\|_{BV}T+\lambda C^{\frac{1}{3}}\|\phi\|_{C_0^\alpha}(\Delta x)^{\alpha+\frac{2}{3}}\|Q_u\|\left(\frac{4XT}{\lambda(\Delta x)^2}\right)^{\frac{2}{3}}.\end{aligned}\tag{5.6.24}$$

Now, with the same arguments leading to (5.6.20), we conclude that the set $\{\tilde{\mathcal{L}}_3^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,q_1}(\Omega)$ for the same $q_1\in(1,2)$ as in the case of $\tilde{\mathcal{L}}_1^\Delta$, $\tilde{\mathcal{L}}_2^\Delta$ and $\tilde{\mathcal{L}}_{4,2}^\Delta$.

By summarizing Steps 1 and 2, we conclude that the collection $\{\tilde{\mathcal{L}}^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,q_1}(\Omega)$, where $q_1\in(1,2)$ is as given in Step 2. Furthermore, combining this result with (5.6.2), we deduce that

$$\{\mathcal{L}^\Delta\}_{\Delta>0}\,\text{is compact in}\,\mathrm{W}^{-1,q}(\Omega),\;\;\text{for}\;\; q=q_1\in(1,2).\tag{5.6.25}$$

Now, owing to the L^∞ boundedness (5.4.3) of the approximate solutions u_Δ (Theorem 5.4.1), it is straightforward to see that $\{\mathcal{L}^\Delta\}$ is bounded in $\mathrm{W}^{-1,r}(\Omega)$ for any $r>2$. Finally, applying the the interpolation result Lemma 5.3.2 we conclude that

$$\{\mathcal{L}^\Delta\}_{\Delta>0}\,\text{is compact in}\,\mathrm{W}^{-1,2}(\Omega).$$

Since Ω is an arbitrary bounded open subset, it follows that $\{\mathcal{L}^\Delta\}_{\Delta>0}$ is compact in $\mathrm{W}^{-1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$. This completes the proof. $\square$

We now present the weak solution convergence result for the proposed scheme (5.2.4), in the following theorem.

Theorem 5.6.2. *Let the initial datum* $u_0 \in (\mathrm{L}^\infty \cap \mathrm{BV})(\mathbb{R})$ *and* $\{u_\Delta\}_{\Delta>0}$ *be the approximate solutions* (5.2.8) *obtained from the second-order scheme* (5.2.4) *under the CFL condition* (5.5.3). *Then, there exists a subsequence* $\{\Delta_m\}_{m\in\mathbb{N}}$ *with* $\lim\limits_{m\to\infty} \Delta_m = 0$ *such that* $\{u_{\Delta_m}\}_{m\in\mathbb{N}}$ *converges strongly to a weak solution of the problem* (5.0.1). *i.e.,*

$$u_{\Delta_m} \xrightarrow{\Delta_m \to 0} u \text{ in } \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+) \text{ for any } p \in [1,\infty) \text{ and a.e. in } \mathbb{R}\times\mathbb{R}_+.$$

Proof. Consider a sequence $\{\Delta_m\}_{m\in\mathbb{N}}$ such that $\lim\limits_{m\to\infty} \Delta_m = 0$. Now, Theorem 5.4.1 and Lemma 5.6.1 allow us to use the compensated compactness result from Theorem 5.3.1 for the sequence $\{u_{\Delta_m}\}_{m\in\mathbb{N}}$. Hence, there exists a subsequence, again denoted by $\{\Delta_m\}_{m\in\mathbb{N}}$, such that $u_{\Delta_m} \to u$ pointwise a.e., as $m \to \infty$. Therefore, for any $p \in [1,\infty)$, it follows that $|u_{\Delta_m} - u|^p \to 0$ as $\Delta_m \to 0$, pointwise a.e.. Since the approximate solutions satisfy the L^∞-estimate $\|u_\Delta\| \leq C_{u_0}$ by Theorem 5.4.1 and the limit $u \in \mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}_+)$ (by Theorem 5.3.1), it follows that $|u_\Delta - u|^p \leq (C_{u_0} + \|u\|)^p$. Consequently, applying the dominated convergence theorem, for any compact set $K \subseteq \mathbb{R}\times\mathbb{R}_+$, we have

$$\begin{aligned} &\lim_{\Delta\to 0} \|u_{\Delta_m} - u\|^p_{\mathrm{L}^p(K)} = 0, \text{ or equivalently,} \\ &u_{\Delta_m} \xrightarrow{\Delta_m \to 0} u \,\text{in}\, \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+) \,\text{for any}\, p \in [1,+\infty). \end{aligned} \tag{5.6.26}$$

Finally, employing a Lax-Wendroff type argument [128] which uses the the L^∞- boundedness of $\{u_\Delta\}_{\Delta>0}$ and the strong convergence $u_{\Delta_m} \to u$ from (5.6.26), we can show that the limit u is a weak solution of (5.0.1). This completes the proof. □

5.7 Convergence to the entropy solution

In order to show the entropy convergence, we follow the approach of [166, 167]. As the first step in this framework, we derive an entropy-convergence result for numerical schemes in the predictor-corrector form:

$$u^{n+1}_{j+\frac{1}{2}} = \bar{u}^{n+1}_{j+\frac{1}{2}} - a^{n+1}_{j+1} + a^{n+1}_j,$$

approximating (5.0.1). This result builds on the idea of Theorem 3.1 in [166], incorporating essential modifications to address the discontinuous flux case, as detailed in the following theorem.

Theorem 5.7.1. *Suppose that a numerical scheme approximating* (5.0.1) *can be written in the form:*

$$u^{n+1}_{j+\frac{1}{2}} = \bar{u}^{n+1}_{j+\frac{1}{2}} - a^{n+1}_{j+1} + a^{n+1}_j, \tag{5.7.1}$$

where

(i) $\{\bar{u}^{n+1}_{j+\frac{1}{2}}\}_{j\in\mathbb{Z}}$ *is computed from* $\{u^n_j\}_{j\in\mathbb{Z}}$ *using the Lax-Friedrichs scheme* (5.2.10) *as:*

$$\bar{u}^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}(u^n_j + u^n_{j+1}) - \lambda\left(f(k_{j+1}, u^n_{j+1}) - f(k_j, u^n_j)\right), \tag{5.7.2}$$

(ii) $|a^{n+1}_j| \le \mathcal{K}\Delta x^\alpha$, $j \in \mathbb{Z}$ *for some constant* $\mathcal{K} > 0$ *which is independent of* Δx *and for some* $\alpha \in (\frac{2}{3}, 1)$,

(iii) The approximate solutions u_Δ *obtained from the predictor-corrector scheme is bounded in the* L^∞*-norm, and for any fixed* $T > 0, X > 0$, *with* $N := \lfloor T/\Delta t\rfloor + 1$ *and* $J := \lfloor X/\Delta x\rfloor + 1$, *satisfies an estimate of the form*

$$\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} |\Delta u^n_{j+\frac{1}{2}}|^3 \le C, \tag{5.7.3}$$

for a constant C *independent of* Δx *and converges pointwise a.e. to a function* $u \in \mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}_+)$.

Then the limit u *of the approximate solutions* u_Δ *generated by the scheme* (5.7.1) *is the entropy solution* (5.1.5) *to the problem* (5.0.1).

Proof. The first-order Lax-Friedrichs scheme (5.2.10) used to obtain $\bar{u}^{n+1}_{j+\frac{1}{2}}$ satisfies a discrete cell entropy inequality (see [116] for more details) given by:

$$\begin{aligned}|\bar{u}^{n+1}_{j+\frac{1}{2}} - c| - \frac{1}{2}|u^n_{j+1} - c| - \frac{1}{2}|u^n_j - c| + \lambda\left(F(k_{j+1}, u^n_{j+1}, c) - F(k_j, u^n_j, c)\right)&\\ - \lambda\,\mathrm{sign}(\bar{u}^{n+1}_{j+\frac{1}{2}} - c)(f(k_{j+1}, c) - f(k_j, c)) \le 0,&\end{aligned}$$

for $F(k, u, c) := \mathrm{sgn}(u - c)(f(k, u) - f(k, c))$. This in turn implies that

$$\begin{aligned}|\bar{u}^{n+1}_{j+\frac{1}{2}} - c| - \frac{1}{2}|u^n_{j+1} - c| - \frac{1}{2}|u^n_j - c| + \lambda\left(F(k_{j+1}, u^n_{j+1}, c) - F(k_j, u^n_j, c)\right)&\\ - \lambda|f(k_{j+1}, c) - f(k_j, c)| \le 0.&\end{aligned} \tag{5.7.4}$$

Now, consider a non-negative test function ϕ with $\mathrm{supp}(\phi) \subseteq [-X, X]\times[0, T]$, let $N = \lfloor T/\Delta t\rfloor + 1$, $J = \lfloor X/\Delta x\rfloor + 1$ and denote $\phi^n_{j+\frac{1}{2}} := \phi(x_{j+\frac{1}{2}}, t^n)$. Adding $|u^{n+1}_{j+\frac{1}{2}} - c|$ to either side of (5.7.4), multiplying it with $\Delta x\phi^n_{j+\frac{1}{2}}$ and summing over $n = 0, 1, \ldots, N-1$ and $j + \frac{n}{2} \in \mathbb{Z}$ with $|j| \le J$, we obtain the following inequality

$$\begin{aligned}\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \phi^n_{j+\frac{1}{2}}|u^{n+1}_{j+\frac{1}{2}} - c| - \frac{1}{2}\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \phi^n_{j+\frac{1}{2}}|u^n_{j+1} - c|&\\ - \frac{1}{2}\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \phi^n_{j+\frac{1}{2}}|u^n_j - c| - \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\le J\\ j+\frac{n}{2}\in\mathbb{Z}}} \phi^n_{j+\frac{1}{2}}\lambda|f(k_{j+1}, c) - f(k_j, c)|&\end{aligned} \tag{5.7.5}$$

$$+\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\phi^n_{j+\frac{1}{2}}\lambda\left(F(k_{j+1},u^n_{j+1},c)-F(k_j,u^n_j,c)\right)\leq\mathcal{J},$$

where

$$\mathcal{J}:=\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\phi^n_{j+\frac{1}{2}}\left(|u^{n+1}_{j+\frac{1}{2}}-c|-|\bar{u}^{n+1}_{j+\frac{1}{2}}-c|\right).$$

First, we show that the term $\mathcal{J}\to 0$ as $\Delta\to 0$. For this, we consider the function $S(u)=\begin{cases}\frac{u^2}{2}+\frac{1}{2}, & \text{if } |u|<1,\\ |u|, & \text{if } |u|\geq 1\end{cases}$ and for a fixed $c\in\mathbb{R}$ and $\epsilon>0$, we define $S_\epsilon(u)=\epsilon S(\dfrac{u-c}{\epsilon})$. It is easy to see that $S_\epsilon\to|u-c|$ as $\epsilon\to 0$, uniformly in the supremum norm. In particular,

$$\begin{aligned} S_\epsilon(u)-|u-c|&=0, \quad &&\text{if } |u-c|\geq\epsilon, \text{ and}\\ \big|S_\epsilon(u)-|u-c|\big|&\leq\frac{3}{2}\epsilon, \quad &&\text{if } |u-c|<\epsilon.\end{aligned} \tag{5.7.6}$$

Upon adding and subtracting suitable terms, $\mathcal{J}$ can be written as

$$\mathcal{J}=\mathcal{J}_0+\mathcal{J}_1+\mathcal{J}_2, \tag{5.7.7}$$

where

$$\mathcal{J}_0:=\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\phi^n_{j+\frac{1}{2}}\left(S_\epsilon(u^{n+1}_{j+\frac{1}{2}})-S_\epsilon(\bar{u}^{n+1}_{j+\frac{1}{2}})\right),$$

$$\mathcal{J}_1:=\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\phi^n_{j+\frac{1}{2}}\left(|u^{n+1}_{j+\frac{1}{2}}-c|-S_\epsilon(u^{n+1}_{j+\frac{1}{2}})\right),$$

$$\mathcal{J}_2:=-\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\phi^n_{j+\frac{1}{2}}\left(|\bar{u}^{n+1}_{j+\frac{1}{2}}-c|-S_\epsilon(\bar{u}^{n+1}_{j+\frac{1}{2}})\right).$$

Now, (5.7.6) implies that for $\epsilon\leq\Delta t^2$,

$$|\mathcal{J}_1|,|\mathcal{J}_2|\leq\frac{3}{2}\epsilon\Delta x\sum_{n=0}^{N-1}\sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}}\|\phi\|\leq\frac{3}{2}\Delta t\|\phi\|(8XT),$$

and hence

$$\lim_{\Delta\to 0}\mathcal{J}_1=\lim_{\Delta\to 0}\mathcal{J}_2=0. \tag{5.7.8}$$

Further, the convexity of S_ϵ gives us

$$S_\epsilon(u^{n+1}_{j+\frac{1}{2}})-S_\epsilon(\bar{u}^{n+1}_{j+\frac{1}{2}})\leq S'_\epsilon(u^{n+1}_{j+\frac{1}{2}})\left(u^{n+1}_{j+\frac{1}{2}}-\bar{u}^{n+1}_{j+\frac{1}{2}}\right)=S'_\epsilon(u^{n+1}_{j+\frac{1}{2}})\left(a^{n+1}_j-a^{n+1}_{j+1}\right). \tag{5.7.9}$$

Using (5.7.9) in $\mathcal{J}_0$ and applying summation by parts, we obtain the following inequality

$$\mathcal{J}_0 \leq \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} a_{j+1}^{n+1} \left(\phi_{j+\frac{3}{2}}^{n} S_\epsilon'(u_{j+\frac{3}{2}}^{n+1}) - \phi_{j+\frac{1}{2}}^{n} S_\epsilon'(u_{j+\frac{1}{2}}^{n+1}) \right). \tag{5.7.10}$$

Now, adding and subtracting the term $\Delta x a_{j+1}^{n+1}\phi_{j+\frac{3}{2}}^{n} S_\epsilon'(u_{j+\frac{1}{2}}^{n+1})$ inside the summation in the RHS of (5.7.10), we have $\mathcal{J}_0 \leq \mathcal{J}_0^a + \mathcal{J}_0^b$, where

$$\mathcal{J}_0^a := \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} a_{j+1}^{n+1}\phi_{j+\frac{3}{2}}^{n} \left(S_\epsilon'(u_{j+\frac{3}{2}}^{n+1}) - S_\epsilon'(u_{j+\frac{1}{2}}^{n+1}) \right),$$

$$\mathcal{J}_0^b := \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} a_{j+1}^{n+1} \left(\phi_{j+\frac{3}{2}}^{n} - \phi_{j+\frac{1}{2}}^{n} \right) S_\epsilon'(u_{j+\frac{1}{2}}^{n+1}).$$

Using the assumption that $|a_{j+1}^{n+1}| \leq \mathcal{K}\Delta x^\alpha$ for $j \in \mathbb{Z}$, along with the condition $\mathrm{supp}(\phi) \subseteq [-X, X] \times [0, T]$, we apply Hölder's inequality, followed by (5.7.3), to obtain the following estimate for $\mathcal{J}_0^a$:

$$\begin{aligned}
|\mathcal{J}_0^a| &\leq \mathcal{K}(\Delta x)^\alpha \|\phi\| \|S_\epsilon''\| \left(\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} |\Delta u_{j+1}^{n+1}|^3 \right)^{\frac{1}{3}} \left(\Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} 1 \right)^{\frac{2}{3}} \\
&\leq \mathcal{K}(\Delta x)^\alpha \|\phi\| \|S_\epsilon''\| C^{\frac{1}{3}} \left(\frac{8XT}{\Delta t} \right)^{\frac{2}{3}} \leq \frac{1}{\lambda^{\frac{2}{3}}} \mathcal{K}(\Delta x)^{\alpha-\frac{2}{3}} \|\phi\| \|S_\epsilon''\| C^{\frac{1}{3}} (8XT)^{\frac{2}{3}}.
\end{aligned}$$

Now, since $\alpha > \frac{2}{3}$ by assumption, it follows that $|\mathcal{J}_0^a| \to 0$ as $\Delta \to 0$. Next, it is straightforward to see that

$$|\mathcal{J}_0^b| = \Big| \frac{1}{\lambda} \Delta x \Delta t \sum_{n=0}^{N-1} \sum_{\substack{|j|\leq J\\ j+\frac{n}{2}\in\mathbb{Z}}} a_{j+1}^{n+1} \left(\frac{\phi_{j+\frac{3}{2}}^{n} - \phi_{j+\frac{1}{2}}^{n}}{\Delta x} \right) S_\epsilon'(u_{j+\frac{1}{2}}^{n+1}) \Big| \leq \frac{1}{\lambda} \mathcal{K}(\Delta x)^\alpha \|\phi_x\| \|S_\epsilon'\| 8XT,$$

by which we conclude that $|\mathcal{J}_0^b| \to 0$ as $\Delta \to 0$. Therefore, $\lim\limits_{\Delta\to 0} \mathcal{J}_0 \leq 0$. This, together with (5.7.8) applied on (5.7.7) imply that

$$\lim_{\Delta\to 0} \mathcal{J} \leq 0. \tag{5.7.11}$$

Finally, using (5.7.11) and proceeding as in the proofs of Lemmas 5.3 and 5.4 from [116], it is easy to show from (5.7.5) that

$$\begin{aligned}
&\int\int_{\mathbb{R}\times\mathbb{R}_+} (|u - c|\phi_t + \mathrm{sign}(u - c)(f(k, u) - f(k, c))\phi_x) \, \mathrm{d}x \, \mathrm{d}t \qquad (5.7.12) \\
&\quad + \int_{\mathbb{R}} |u_0 - c|\phi(x, 0) \, \mathrm{d}x + \int\int_{(\mathbb{R}\setminus D)\times\mathbb{R}_+} |f(k(x), c)_x|\phi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

$$+\sum_{m=1}^{M}\int_0^\infty |f(k_m^+,c)-f(k_m^-,c)|\phi(x_m,t)\ \mathrm{d}t \geq 0,$$

where D is as in **H6**. This concludes the proof. □

Remark 5.7.2. We emphasize that the entropy convergence result in Theorem 3.1 of [166] is based on a BV-estimate for the approximate solutions. However, for conservation laws with discontinuous coefficients, BV-estimates need not be available in general (see [3]). In this context, as a key novelty of our approach, we show in Theorem 5.7.1 that a significantly weaker cubic estimate of the form (5.5.4) is sufficient to establish the desired entropy convergence.

Now, our strategy essentially is to use Theorem 5.7.1 to prove the convergence of the proposed second-order scheme (5.2.4) to the entropy solution. To this end, we note that the scheme (5.2.4) can be written in the predictor-corrector form

$$\begin{aligned}\bar{u}^{n+1}_{j+\frac{1}{2}} &= \frac{1}{2}(u_j^n+u_{j+1}^n)-\lambda\left(f(k_{j+1},u_{j+1}^n)-f(k_j,u_j^n)\right),\\ u^{n+1}_{j+\frac{1}{2}} &= \bar{u}^{n+1}_{j+\frac{1}{2}}-a_{j+1}^{n+1}+a_j^{n+1},\end{aligned}\tag{5.7.13}$$

with the correction terms

$$a_j^{n+1} := \lambda\left(f(k_j,u_j^{n+\frac{1}{2}})-f(k_j,u_j^n)\right)+\frac{1}{8}\sigma_j^n = \left(-\frac{\lambda^2}{2}f_u(k_j,\zeta_j)f_u(k_j,u_j^n)+\frac{1}{8}\right)\sigma_j^n,\tag{5.7.14}$$

for some $\zeta_j \in \mathcal{I}(u_j^n,u_j^{n+\frac{1}{2}})$. Next, we modify the scheme (5.7.13) by redefining the slopes (5.2.2) as

$$\sigma_j^n = \operatorname{minmod}\left((u_{j+1}^n-u_j^n),\frac{1}{2}(u_{j+1}^n-u_{j-1}^n),(u_j^n-u_{j-1}^n),\operatorname{sign}(u_{j+1}^n-u_j^n)\tilde{\mathcal{K}}(\Delta x)^\alpha\right),\tag{5.7.15}$$

for some constants $\tilde{\mathcal{K}}>0$ and $\alpha\in(\frac{2}{3},1)$. With this modification, it is direct to see from (5.7.14) that the correction terms $\{a_j^{n+1}\}_{j\in\mathbb{Z}}$ in (5.7.13) satisfy the estimate

$$|a_j^{n+1}| \leq \mathcal{K}(\Delta x)^\alpha,\tag{5.7.16}$$

for $j\in\mathbb{Z}$, where $\mathcal{K} := \left(\frac{1}{2}\lambda^2\|f_u\|^2+\frac{1}{8}\right)\tilde{\mathcal{K}}$. Finally, we conclude this section with the entropy convergence result, stated below.

Theorem 5.7.3. *Let the initial datum u_0 be such that $u_0\in(\mathrm{L}^\infty\cap\mathrm{BV})(\mathbb{R})$, with $\underline{u}\leq u_0(x)\leq\bar{u}$, for $x\in\mathbb{R}$. Under the CFL condition* (5.5.3) *and hypotheses **H1-H7**, the approximate solutions $\{u_\Delta\}_{\Delta>0}$* (5.2.8) *obtained from the scheme* (5.2.4) *with the modified slopes* (5.7.15) *converge to the entropy solution* (5.1.5) *of the problem* (5.0.1).

Proof. Since the second-order scheme (5.2.4) with the modified slopes (5.7.15) can be reformulated in the predictor-corrector form, it suffices to show that the hypotheses (i)-(iii) of Theorem 5.7.1 hold true. From (5.7.13), it is clear that the predictor step employs the Lax-Friedrichs time-stepping, thereby confirming the validity of condition (i) in Theorem 5.7.1. From (5.7.16), it follows that with the slope modification (5.7.15), the hypothesis (ii) of Theorem 5.7.1 also holds. Finally, observing that the L^∞- estimate (5.4.3) (Theorem 5.4.1), the cubic estimate (5.5.4) (Lemma 5.5.2), the $\mathrm{W}^{-1,2}_{\mathrm{loc}}$ compactness result (Lemma 5.6.1), and the convergence theorem (Theorem 5.6.2) remain valid with the modified slopes (5.7.15), it follows that hypothesis (iii) is also satisfied. Now, an application of Theorem 5.7.1 yields the desired convergence to the entropy solution. □

Remark 5.7.4. In the implementation of the numerical scheme, the slope modification (5.7.15) is not really required. This is because, for any given mesh-size Δx, we can choose $\tilde{K} > 0$ large enough so that the modified slope (5.7.15) reduces to (5.2.2). In particular, for mesh sizes $\Delta x \geq \epsilon$ for some fixed $\epsilon > 0$, we can choose $\tilde{K} = 2C_{u_0}\epsilon^{-\alpha}$, where C_{u_0} is as given in (5.4.3). For more details, see [131] (p. 158, below Eq. (26)), [166] (p. 68, below Fig. 3) and [165] (p. 577, Remark).

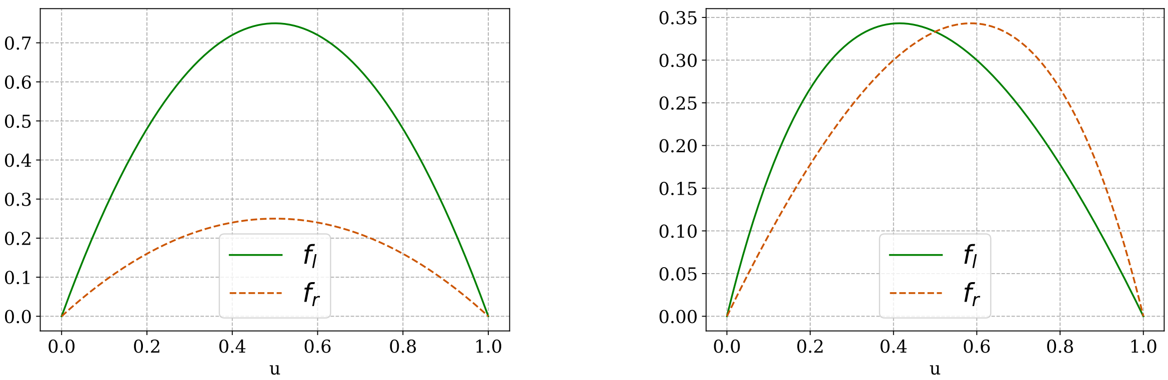
5.8 Numerical experiments

In this section, we present numerical experiments to illustrate the performance of the second-order scheme (5.2.4) in comparison to the first-order Lax-Friedrichs scheme (5.2.10). In the following, we denote the second-order scheme (5.2.4) by SO and the Lax-Friedrichs scheme (5.2.10) by LF. Alluding to Remark 5.1.1, we focus on examples where the flux function is strictly concave, as these are the types of fluxes most commonly considered in the literature. We divide the computational domain into cells of size Δx. The CFL condition (5.5.1), required for proving the convergence of the SO scheme, is highly restrictive and need not be optimal. Therefore, to compute the time-step Δt, we impose the less restrictive CFL condition (5.4.1) (which ensures the maximum principle). In each test case, both the LF and SO solutions are computed with the time-step corresponding to the SO scheme. Also, we apply absorbing boundary conditions in both the examples.
Example 5.1. We consider an example from [112], where the flux function in (5.0.1) is given by:

$$f(k,u) = ku(1-u), \quad k(x) \;= \begin{cases} 3 & \text{if}\, x < 0, \\ 1 & \text{if}\, x \geq 0. \end{cases} \tag{5.8.1}$$

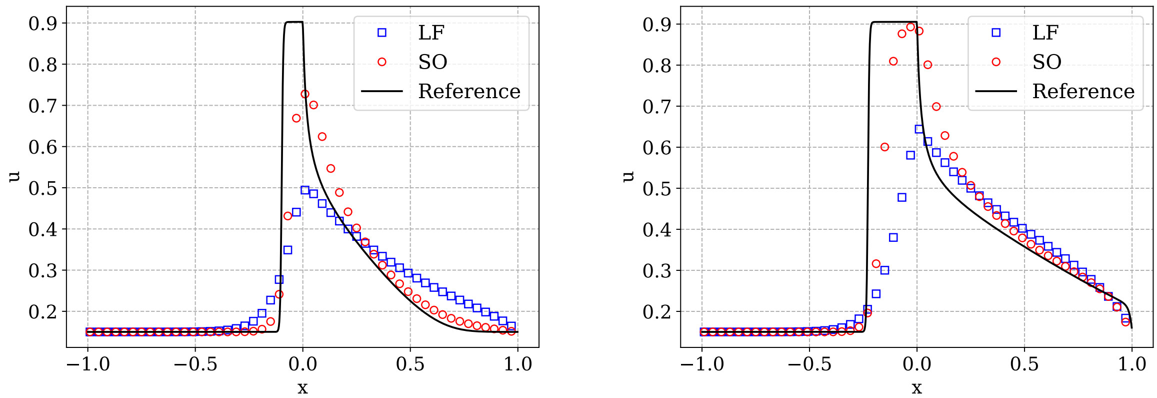
The discontinuous flux (5.8.1) is depicted in Figure 5.1(a), where $f_l(u) = 3u(1-u)$ and $f_r(u) = u(1-u)$.

Note that there is no flux crossing in this case and the crossing condition **H7** holds



(a) Example 5.1 (b) Example 5.2

Figure 5.1: The fluxes to the left (f_l) and right (f_r) of $x = 0$.



(a) $t = 0.8$ (b) $t = 1.6$

Figure 5.2: Example 5.1. Numerical solutions obtained by evolving (5.8.2) with $\Delta x = 0.04$ and $\Delta t = \frac{1}{750}$.

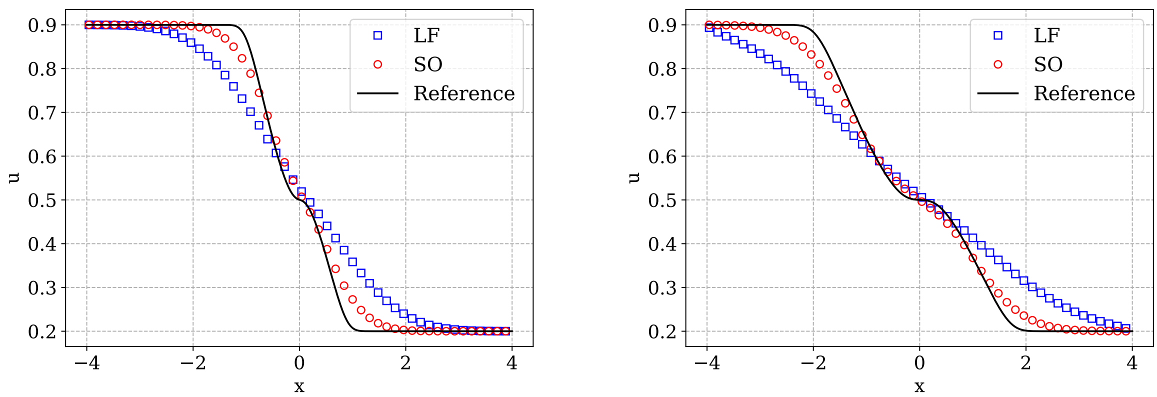
trivially for (5.8.1). We set the initial condition as the constant function

$$u_0(x) = 0.15, \tag{5.8.2}$$

and compute the numerical solutions in the domain $[-1, 1]$ with a mesh of size $\Delta x = 2/50$ at the time levels $t \in \{0.8, 1.6\}$. Here, the reference solutions are computed with the LF scheme using a fine mesh of size $\Delta x = 2/1000$. The results are displayed in Figure 5.2. We observe that the SO scheme exhibits lower numerical diffusion and provides a more accurate approximation of the solution, especially near the discontinuities.

Example 5.2. Next, we consider an example studied in [134], where the flux function in (5.0.1) is of the form

$$f(H(x), u(x,t)) = H(x)f_r(u) + (1 - H(x))f_l(u) = \begin{cases} f_l(u) & \text{for } x < 0, \\ f_r(u) & \text{for } x \geq 0, \end{cases} \tag{5.8.3}$$



(a) $t = 1.0$ (b) $t = 2.0$

Figure 5.3: Example 5.2. Numerical solutions obtained by evolving (5.8.5) with $\Delta x = 0.16$ and $\Delta t = 0.008$.

with

$$f_l(u) := \frac{2u(1-u)}{1+u}, \quad f_r(u) := \frac{2u(1-u)}{2-u}. \tag{5.8.4}$$

The discontinuous flux (5.8.3) is depicted in Figure 5.1(b), where we note that the crossing condition **H7** is satisfied by (5.8.3). We set the initial datum to be

$$u(x,0) = \begin{cases} 0.9 & \text{if } x \leq 0, \\ 0.2 & \text{otherwise.} \end{cases} \tag{5.8.5}$$

and compute the solutions using $\Delta x = 8/50$ at the time levels $t = 1.0$ and $t = 2.0$, which are compared in Figures 5.3(a) and 5.3(b), respectively. The reference solutions are obtained with the LF scheme on a fine mesh of size $\Delta x = 8/2000$. As shown in Figure 5.3, the proposed SO scheme yields more accurate solutions, in comparison to the LF scheme.

5.9 Concluding remarks

In this work, we have proposed and analyzed a second-order numerical scheme based on MUSCL-type reconstruction for conservation laws with discontinuous flux functions, where the flux may exhibit multiple discontinuities. The scheme features a simple formulation inspired by the Nessyahu-Tadmor central scheme. We employ the framework of compensated compactness to establish the convergence of the proposed scheme. This analysis requires several key estimates, which we carefully derive by exploiting the structural properties of the numerical method.

The uniqueness of solutions to conservation laws with discontinuous flux is a topic of considerable interest, and various approaches exist in the literature. In this chapter, we focus on a Kruzkov-type entropy condition equivalent to the vanishing viscosity solution approach. We establish the convergence of entropy solutions by incorporating a mesh-dependent parameter into the scheme. To the best of our knowledge, this is the first comprehensive convergence result, including entropy solutions, for a MUSCL-type reconstruction scheme applied to conservation laws with discontinuous flux. The numerical solutions presented in Section 5.8, where we compare the proposed second-order scheme and the first-order Lax-Friedrichs (LF) scheme, clearly indicate the superiority of the SO scheme over its FO counterpart.

6

Conclusion

In this thesis, we studied the design, analysis, and numerical validation of second-order accurate schemes for a broad class of conservation laws with spatially dependent, non-local or discontinuous flux functions. A key objective was to address the analytical and computational challenges introduced by the non-locality and flux discontinuities, and to develop high-order discretizations that remain robust in such settings. To this end, we developed second-order schemes based on MUSCL-type spatial reconstructions, coupled either with multi-stage time-stepping methods like Runge–Kutta or single-stage techniques such as MUSCL–Hancock and Nessyahu–Tadmor. Particular care was taken to handle the non-local or discontinuous dependencies in the flux function, ensuring that the schemes not only provide formal second-order accuracy but also admit rigorous theoretical results.

In Chapter 2, we developed second-order MUSCL-type schemes for scalar non-local conservation laws arising in traffic flow modeling. Two time-discretization strategies were proposed: a Runge–Kutta (RK-2) method and a single-stage MUSCL–Hancock (MH) variant. For the RK-2 scheme, convergence to the unique entropy solution was established through a sequence of estimates, an application of Kolmogorov's compactness theorem, and a mesh-dependent modification of the minmod limiter. Both schemes demonstrated improved accuracy and stability over first-order methods. Notably, numerical results showed that the MH scheme performed slightly better in terms of computational efficiency, although a theoretical convergence proof for it remains elusive.

In Chapter 3, we focused on a broader class of non-local conservation laws with a general (possibly non-monotone) convolution kernel and a general flux function. We proposed a single-stage MUSCL Hancock (MH)-type second-order scheme and rigorously established its convergence. The construction of the scheme involved a careful treatment of discrete convolutions, in both the predictor and corrector stages of the MH scheme, which was crucial both for second-order accuracy and for enabling theoretical convergence analysis. Numerical experiments confirmed the improved accuracy of the MH scheme compared to first-order methods and also demonstrated its superior computational efficiency relative to Runge Kutta-based two-stage schemes.

In Chapter 4, for systems of non-local conservation laws in multiple space dimensions, we proposed a fully discrete second-order (SO) scheme. We proved that the scheme is positivity-preserving and L^∞-stable. Numerical results supported the theoretical findings and demonstrated that the SO scheme offers significantly improved accuracy and robustness over a corresponding first-order (FO) scheme. Additionally, in the context of the so-called singular limit problem, the SO scheme showed a stronger convergence to the local model as the non-local interaction parameter r tends to zero, outperforming the FO scheme.

In Chapter 5, we established the theoretical convergence of a second-order MUSCL-type scheme for a general class of scalar conservation laws with discontinuous flux functions. The scheme is inspired by the Nessyahu–Tadmor central method and is analyzed within the framework of compensated compactness. Several essential estimates, including a cubic estimate, are derived by carefully exploiting the structure of the numerical method. These estimates serve as critical components in the compactness argument. Adopting a Kružkov-type entropy condition, we incorporate a mesh-dependent parameter in the slope reconstruction step to ensure convergence of the proposed scheme to the entropy solution. Furthermore, we prove that the cubic estimate, although weaker than a bounded variation estimate, is sufficient in the entropy convergence, thereby generalizing the entropy convergence framework of [166, 167]. Numerical experiments support the theoretical findings and confirm the superior accuracy of the proposed second-order scheme compared to the first-order Lax–Friedrichs method.

A

Appendix A

A.1 Proof of Theorem 2.3.1

Proof. By Theorem A.8 of [108]), for each fixed $t \in [0, T]$ and for any sequence $\xi_j \to 0$ there exists a subsequence, again denoted by ξ_j, such that $\{u_{\xi_j}(t)\}$ converges to a function $u(t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$.

Now, consider a countable dense subset E of the interval $[0, T]$. By a diagonalization argument, we can extract again a subsequence (still denoted by ξ_j) such that

$$\int_B |u_{\xi_j}(t,x) - u(t,x)| \, \mathrm{d}x \to 0 \text{ as } \xi_j \to 0, \text{ for } t \in E. \tag{A.1.1}$$

Let $\epsilon > 0$ be given. Then there exists a positive δ such that $\omega_T^B \tilde{\delta} \le \epsilon$ for all $\tilde{\delta} \le \delta$. Fix $t \in [0, T]$. By the denseness of E, there exists a $t_k \in E$ with $|t_k - t| \le \delta$. Therefore, by (2.3.3)

$$\begin{aligned} \int_B |u_{\tilde{\xi}}(t,x) - u(t_k,x)| \, \mathrm{d}x &\le \omega_T^B(|t - t_k|) + \mathcal{O}(\tilde{\xi}) \\ &\le \epsilon + \mathcal{O}(\tilde{\xi}) \text{ for } \tilde{\xi} \le \xi \end{aligned}$$

and by (A.1.1)

$$\int_B |u_{\xi_{j_1}}(t_k,x) - u_{\xi_{j_2}}(t_k,x)| \, \mathrm{d}x \le \epsilon \text{ for } \xi_{j_1}, \xi_{j_2} \le \xi \text{ and } t_k \in E.$$

Further, applying the triangle inequality, it yields

$$
\begin{aligned}
&\int_B |u_{\xi_{j_1}}(t,x) - u_{\xi_{j_2}}(t,x)| \,\mathrm{d}x \\
&\quad \leq \int_B |u_{\xi_{j_1}}(t,x) - u_{\xi_{j_1}}(t_k,x)| \,\mathrm{d}x + \int_B |u_{\xi_{j_1}}(t_k,x) - u_{\xi_{j_2}}(t_k,x)| \,\mathrm{d}x \\
&\quad + \int_B |u_{\xi_{j_2}}(t_k,x) - u_{\xi_{j_2}}(t,x)| \,\mathrm{d}x \\
&\quad \leq \omega_T^B(|t-t_k|) + \mathcal{O}(\xi_{j_1}) + \epsilon + \omega_T^B(|t-t_k|) + \mathcal{O}(\xi_{j_2}) \\
&\quad \leq 3\epsilon + \mathcal{O}(\xi).
\end{aligned}
$$

This shows that $u_\xi(t) \to u(t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ for each $t \in [0,T]$. Finally, by the dominated convergence theorem it follows that

$$
\sup_{t\in[0,T]} \int_B |u_\xi(t,x) - u(t,x)| \,\mathrm{d}x \text{ as } \xi \to 0.
$$

This completes the proof. □

A.2 Mean downstream velocity model

Now, we consider the mean downstream velocity model of non-local traffic proposed in [89]. The corresponding model is given by

$$
\begin{aligned}
\partial_t \rho + \partial_x \Big(g(\rho)\big(v(\rho) * w_\eta\big) \Big) &= 0, \quad x \in \mathbb{R}, t \in (0,T], \\
\rho(x,0) &= \rho_0(x), \qquad x \in \mathbb{R},
\end{aligned}
\tag{A.2.1}
$$

where $v(\rho) * w_\eta(t,x) = \int_x^{x+\eta} v(\rho(t,y)) w_\eta(y-x)\,\mathrm{d}y$ and the terms g, v are as in Section 2.1. This model assumes that the drivers adapt their velocity by evaluating the average velocity of vehicles in a neighbourhood in front of them, giving greater importance to closer vehicles.

A.2.1 Second-order scheme

Here we extend the second-order scheme considered in Section 2.2 to approximate (A.2.1). We proceed as in Section 2.2, where the main difference is in evaluating the convolution term. Now, the second-order scheme is written as

$$
\begin{aligned}
\rho_j^{(1)} &= \rho_j^n - \lambda \left(g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right), \\
\rho_j^{(2)} &= \rho_j^{(1)} - \lambda \left(g(\rho^{(1)}_{j+\frac{1}{2},-}) V^{(1)}_{j+\frac{1}{2}} - g(\rho^{(1)}_{j-\frac{1}{2},-}) V^{(1)}_{j-\frac{1}{2}} \right), \\
\rho_j^{n+1} &= \frac{1}{2} \left(\rho_j^n + \rho_j^{(2)} \right),
\end{aligned}
$$

where $V^n_{j+\frac{1}{2}}$ and $V^{(1)}_{j+\frac{1}{2}}$ are the approximations of the convolution term $v(\rho) * w_\eta$ at the respective Runge-Kutta time steps, which are given by

$$V^n_{j+\frac{1}{2}} := \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(v(\rho^n_{j+k+\frac{1}{2},+})w^k_\eta + v(\rho^n_{j+k+\frac{3}{2},-})w^{k+1}_\eta\right), \tag{A.2.2}$$

$$V^{(1)}_{j+\frac{1}{2}} := \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(v(\rho^{(1)}_{j+k+\frac{1}{2},+})w^k_\eta + v(\rho^{(1)}_{j+k+\frac{3}{2},-})w^{k+1}_\eta\right). \tag{A.2.3}$$

Remark A.2.1. If the quadrature rule used in (A.2.2) and (A.2.3) is not exact for the given kernel function, (i.e., if $\frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w^k_\eta + w^{k+1}_\eta\right) \neq 1$), then we replace w^k_η by $\tilde{w}^k_\eta = \frac{w^k_\eta}{Q_{\Delta x}}$ in (A.2.2) and (A.2.3), where $Q_{\Delta x} := \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w^k_\eta + w^{k+1}_\eta\right)$.

A.2.2 MUSCL-Hancock type scheme

To propose a MUSCL-Hancock scheme for the model (A.2.1), we proceed exactly as in Section 2.5, where $v(R^n_{j+\frac{1}{2},\pm})$ and $v(R^{n+\frac{1}{2}}_{j+\frac{1}{2}})$ in (2.5.1) and (2.5.2) are replaced by $V^n_{j+\frac{1}{2},\pm}$ and $V^{n+\frac{1}{2}}_{j+\frac{1}{2}}$, respectively. These terms are defined as follows

$$V^n_{j+\frac{1}{2},-} := \Delta x\sum_{k=0}^{N-1} v(\rho^n_{j+k+\frac{1}{2},-})w^k_\eta, \tag{A.2.4}$$

$$V^n_{j+\frac{1}{2},+} := \Delta x\sum_{k=0}^{N-1} v(\rho^n_{j+k+\frac{1}{2},+})w^k_\eta, \tag{A.2.5}$$

$$V^{n+\frac{1}{2}}_{j+\frac{1}{2}} := \sum_{k=0}^{N-1}\gamma_k v(\rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+}) + \frac{1}{\Delta x}\sum_{k=0}^{N-1}\chi_k\left(v(\rho^{n+\frac{1}{2}}_{j+k+\frac{3}{2},-}) - v(\rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+})\right).$$

Remark A.2.2. If the quadrature rule used in (A.2.4) and (A.2.5) is not exact for the given kernel function (i.e., if $\Delta x\sum_{k=0}^{N-1} w^k_\eta \neq 1$), then we replace w^k_η by $\tilde{w}^k_\eta := \frac{w^k_\eta}{Q_{\Delta x}}$ in (A.2.4) and (A.2.5), where $Q_{\Delta x} := \Delta x\sum_{k=0}^{N-1} w^k_\eta$.

A.3 Results for the FSST scheme

We state a technical lemma (Lemma A.1, page 32, [32]) without proof, which will be used in the next lemma.

Lemma A.3.1. *Consider a sequence of functions $\psi_{\Delta x} : \mathbb{R} \to \mathbb{R}$ such that there exists a uniform bound $\|\psi_{\Delta x}\| \le C$. Also assume that $\psi_{\Delta x}(x)$ converges to a function ψ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. Then for all $\zeta \in \mathbb{R}$, $\psi_{\Delta x}(x + \zeta\Delta x)$ converges to $\psi(x)$ as $\Delta x \to 0$ for a.e. $x \in \mathbb{R}$.*

Let us recall the notations $\rho_{\Delta x}(t,x) := \rho_j^n$, and $\rho_{\Delta x}^{(l)}(t,x) := \rho_j^{(l)}$ for $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $l = 1,2$ with $\rho_j^{(l)}$ computed from ρ_j^n for all $j \in \mathbb{Z}$. Now, we present a lemma which is used in the proof of Theorem 2.4.4.

Lemma A.3.2. *Assume that $\rho_{\Delta x}$ and $\rho_{\Delta x}^{(l)}$, $l = 1,2$ obtained from* (2.4.2) *are uniformly bounded and that $\rho_{\Delta x}$ converges to a function ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$. Then $\rho_{\Delta x}^{(l)}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$ for $l = 1,2$.*

Proof. From the first time-step in (2.4.2), we can write

$$\rho_{\Delta x}^{(1)}(t,x) = \rho_{\Delta x}(t,x) - \lambda\left(f_{\Delta x}\big(t, x + \frac{\Delta x}{2}\big) - f_{\Delta x}\big(t, x - \frac{\Delta x}{2}\big)\right), \tag{A.3.1}$$

where $f_{\Delta x}(t,x) := g(\rho_j^n)V_{j+\frac{1}{2}}^n$ for $(t,x) \in [t^n, t^{n+1}) \times (x_j, x_{j+1}]$. Clearly

$$f_{\Delta x}(t, x + \frac{\Delta x}{2}) = g\left(\rho_{\Delta x}(t,x)\right) v\left(R_{\Delta x}(t, x + \Delta x)\right),$$

where

$$R_{\Delta x}(t,x) := R_{j-\frac{1}{2}}^n \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}).$$

Note that

$$\begin{aligned} &R_{\Delta x}(t, x + \Delta x) \\ &\quad = \frac{1}{2}\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta} \rho_{\Delta x}(t, y + \Delta x)\left(w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + \Delta x) + w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + 2\Delta x)\right) \mathrm{d}y, \end{aligned}$$

for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, where $w_{\eta,\Delta x}(x) := w_\eta^k$ for $x \in (k\Delta x, (k+1)\Delta x]$ and $w_{\eta,\Delta x}(0) := w_\eta(0)$. Using Lemma A.3.1 and employing the dominated convergence theorem, we can conclude that the term $R_{\Delta x}(t, x + \Delta x)$ converges to $\int_x^{x+\eta} \rho(t,y) w_\eta(y - x)\,\mathrm{d}y$ as $\Delta x \to 0$. Now, using the continuity of g and v, we can conclude that $f_{\Delta x}\big(t, x + \frac{\Delta x}{2}\big)$ converges to $g\big(\rho(t,x)\big)v\big(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y\big)$ *a.e.* Similarly, it can be shown that $f_{\Delta x}\big(t, x - \frac{\Delta x}{2}\big)$ also converges to $g(\rho(t,x))v\big(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y\big)$ *a.e.* Thus, taking the limit $\Delta x \to 0$ in (A.3.1), we conclude that $\rho_{\Delta x}^{(1)}$ converges to ρ *a.e.* Additionally, as $\rho_{\Delta x}^{(1)}$ is uniformly bounded by hypothesis, the dominated convergence theorem implies that $\rho_{\Delta x}^{(1)}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$. Following similar arguments, we can show that $\rho_{\Delta x}^{(2)}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R})$. □

B

Appendix B

B.1 Technical estimates for the total variation bound

In this section, we derive certain estimates required in the proof of Theorem 3.3.8 from Chapter 3.

B.1.1 Bound on $\sum_{j\in\mathbb{Z}}|\tilde{C}^n_{j+\frac{1}{2}}|$

In order to estimate the term $\sum\limits_{j\in\mathbb{Z}}|\tilde{C}^n_{j+\frac{1}{2}}|$, we first show that $0 \leq \tilde{k}^n_{j+\frac{1}{2}}, \tilde{\ell}^n_{j+\frac{1}{2}} \leq \dfrac{1}{2}$, for $j \in \mathbb{Z}$. Using Remark 3.3.1 (equation (3.3.2)) and the CFL condition (3.3.12), it is straightforward to see that

$$\begin{aligned}
&\left(\frac{\rho^{n,-}_{j+\frac{3}{2}} - \rho^{n,-}_{j+\frac{1}{2}}}{\rho^n_{j+1} - \rho^n_j}\right)\left(1 - \frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,-}_{j+1}, A^{n,-}_{j+\frac{3}{2}})\right) + \left(\frac{\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}}{\rho^n_{j+1} - \rho^n_j}\right)\left(\frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,+}_j, A^{n,+}_{j+\frac{1}{2}})\right)\\
&\quad \geq \frac{1}{2}\left(1 - \frac{\lambda}{2}\|\partial_\rho f\|\right) - (1+\theta)\frac{\lambda}{2}\|\partial_\rho f\| \geq 0.
\end{aligned} \tag{B.1.1}$$

Further, from the CFL condition (3.3.12), it follows that $\dfrac{1}{2}\left(\lambda\partial_\rho f(\bar{\rho}^{n+\frac{1}{2},-}_{j+1}, A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) + \alpha\right) \geq 0$, which together with (B.1.1) implies that $\tilde{k}^n_{j+\frac{1}{2}} \geq 0$. Using the CFL condition (3.3.12) once again, we obtain

$$\tilde{k}^n_{j+\frac{1}{2}} \leq \frac{1}{2}\left(\lambda\|\partial_\rho f\| + \alpha\right)(1+\theta)\left(1 + \lambda\|\partial_\rho f\|\right) \leq \frac{1}{2} \times \frac{8}{27} \times \frac{3}{2} \times \frac{29}{27} \leq \frac{1}{2}, \tag{B.1.2}$$

thereby concluding that $0 \leq \tilde{k}^n_{j+\frac{1}{2}} \leq \frac{1}{2}$, for all $j \in \mathbb{Z}$. Analogously, we obtain a bound $0 \leq \tilde{\ell}^n_{j+\frac{1}{2}} \leq \frac{1}{2}$.

Since $0 \leq \tilde{k}^n_{j+\frac{1}{2}}, \tilde{\ell}^n_{j+\frac{1}{2}} \leq \frac{1}{2}$ for $j \in \mathbb{Z}$, from (3.3.62) we obtain the desired estimate

$$\sum_{j\in\mathbb{Z}} |\tilde{C}^n_{j+\frac{1}{2}}| \leq \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|. \tag{B.1.3}$$

B.1.2 Bound on $\sum_{j\in\mathbb{Z}} |\hat{C}^n_{j+\frac{1}{2}}|$

Recall from (3.3.62) that

$$\hat{C}_{j+\frac{1}{2}} = -\hat{\ell}^n_{j+\frac{1}{2}} - \hat{k}^n_{j+\frac{1}{2}} + \hat{\ell}^n_{j+\frac{3}{2}} + \hat{k}^n_{j-\frac{1}{2}}.$$

To analyze this term, we first focus on the difference $\hat{\ell}^n_{j+\frac{1}{2}} - \hat{\ell}^n_{j-\frac{1}{2}}$. Upon expanding it using the definition in (3.3.60) and subsequently rearranging the terms, we obtain

$$\hat{\ell}^n_{j+\frac{1}{2}} - \hat{\ell}^n_{j-\frac{1}{2}} = \mathcal{L}^1_j + \mathcal{L}^2_j, \tag{B.1.4}$$

where

$$\begin{aligned}
\mathcal{L}^1_j :=& \frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left[-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_{j+1})\left(A^{n,-}_{j+\frac{3}{2}} - A^{n,-}_{j+\frac{1}{2}}\right)\right]\\
&- \frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_{j-1}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\left[-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j-\frac{1}{2}}, \bar{A}^{n,-}_{j})\left(A^{n,-}_{j+\frac{1}{2}} - A^{n,-}_{j-\frac{1}{2}}\right)\right],\\
\mathcal{L}^j_2 :=& \frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left[\frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac{1}{2}}, \bar{A}^{n,+}_{j})\left(A^{n,+}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}}\right)\right]\\
&- \frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_{j-1}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\left[\frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac{3}{2}}, \bar{A}^{n,+}_{j-1})\left(A^{n,+}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{3}{2}}\right)\right].
\end{aligned} \tag{B.1.5}$$

Turning to the term $\mathcal{L}^1_j$, by adding and subtracting suitable terms, we write

$$\begin{aligned}
\mathcal{L}^1_j =& \frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left(-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_{j+1})\right)\left(A^{n,-}_{j+\frac{3}{2}} - 2A^{n,-}_{j+\frac{1}{2}} + A^{n,-}_{j-\frac{1}{2}}\right)\\
&+ \left(A^{n,-}_{j+\frac{1}{2}} - A^{n,-}_{j-\frac{1}{2}}\right)\left[\frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_j, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\left(-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_{j+1})\right)\right.\\
&\left.-\frac{1}{2}\left(\alpha - \lambda \partial_\rho f(\bar{\rho}^{n+\frac{1}{2},+}_{j-1}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\left(-\frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j-\frac{1}{2}}, \bar{A}^{n,-}_{j})\right)\right].
\end{aligned} \tag{B.1.6}$$

Further, appropriately adding and subtracting the terms $\mathcal{T}_1 := \partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_j)$, $\mathcal{T}_2 :=$

$\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})$ and $\mathcal{T}_3 := \partial_A f(\rho_{j-\frac{1}{2}}^{n,-}, \bar{A}_j^{n,-})$ to (B.1.6), we obtain

$$
\begin{aligned}
\mathcal{L}_j^1 = &\frac{1}{2}\beta(n,j)\left(-\frac{\lambda}{2}\partial_A f(\rho_{j+\frac{1}{2}}^{n,-}, \bar{A}_{j+1}^{n,-})\right)\left(A_{j+\frac{3}{2}}^{n,-} - 2A_{j+\frac{1}{2}}^{n,-} + A_{j-\frac{1}{2}}^{n,-}\right)\\
&+ \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right)\frac{1}{2}\beta(n,j)\frac{\lambda}{2}\left(-\partial_A f(\rho_{j+\frac{1}{2}}^{n,-}, \bar{A}_{j+1}^{n,-}) + \mathcal{T}_1 - \mathcal{T}_1 + \mathcal{T}_3\right)\\
&+ \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right)\left(-\frac{\lambda}{2}\mathcal{T}_3\right)\frac{1}{2}\left[\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right)\right.\\
&\left.+\lambda\mathcal{T}_2 - \lambda\mathcal{T}_2 - \left(\alpha - \lambda\partial_\rho f(\bar{\rho}_{j-1}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})\right)\right],
\end{aligned}
\tag{B.1.7}
$$

where we define $\beta(n,j) := \alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})$. Now, we split $\mathcal{L}_j^1$ by applying the mean value theorem as

$$
\mathcal{L}_j^1 = \mathcal{L}_j^a + \mathcal{L}_j^b + \mathcal{L}_j^c + \mathcal{L}_j^d,
\tag{B.1.8}
$$

where

$$
\begin{aligned}
\mathcal{L}_j^a &:= \frac{1}{2}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right)\left(-\frac{\lambda}{2}\partial_A f(\rho_{j+\frac{1}{2}}^{n,-}, \bar{A}_{j+1}^{n,-})\right)\left(A_{j+\frac{3}{2}}^{n,-} - 2A_{j+\frac{1}{2}}^{n,-} + A_{j-\frac{1}{2}}^{n,-}\right),\\
\mathcal{L}_j^b &:= \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right)\frac{1}{2}\left(\alpha - \lambda\partial_\rho f(\bar{\rho}_j^{n+\frac{1}{2},+}, A_{j+\frac{1}{2}}^{n+\frac{1}{2}})\right)\frac{\lambda}{2}\left(\partial_{\rho A}^2 f(\bar{\rho}_j^{n,-}, \bar{A}_j^{n,-})(\rho_{j-\frac{1}{2}}^{n,-} - \rho_{j+\frac{1}{2}}^{n,-})\right.\\
&\quad\left.+ \partial_{AA}^2 f(\rho_{j+\frac{1}{2}}^{n,-}, \bar{A}_{j+\frac{1}{2}}^{n,-})(\bar{A}_j^{n,-} - \bar{A}_{j+1}^{n,-})\right),\\
\mathcal{L}_j^c &:= \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right)\left(-\frac{\lambda}{2}\mathcal{T}_3\right)\frac{\lambda}{2}\partial_{\rho\rho}^2 f(\bar{\bar{\rho}}_{j-\frac{1}{2}}^{n+\frac{1}{2},+}, A_{j-\frac{1}{2}}^{n+\frac{1}{2}})(\bar{\rho}_{j-1}^{n+\frac{1}{2},+} - \bar{\rho}_j^{n+\frac{1}{2},+}),\\
\mathcal{L}_j^d &:= \left(A_{j+\frac{1}{2}}^{n,-} - A_{j-\frac{1}{2}}^{n,-}\right)\left(-\frac{\lambda}{2}\mathcal{T}_3\right)\frac{\lambda}{2}\partial_{\rho A}^2 f(\bar{\rho}_j^{n+\frac{1}{2},+}, \bar{A}_j^{n+\frac{1}{2}})(A_{j-\frac{1}{2}}^{n+\frac{1}{2}} - A_{j+\frac{1}{2}}^{n+\frac{1}{2}}),
\end{aligned}
\tag{B.1.9}
$$

where $\bar{\rho}_j^{n+\frac{1}{2},\pm} \in \mathcal{I}(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},\pm}, \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},\pm})$, $\bar{\bar{\rho}}_{j-\frac{1}{2}}^{n+\frac{1}{2},+} \in \mathcal{I}(\bar{\rho}_{j-1}^{n+\frac{1}{2},+}, \bar{\rho}_j^{n+\frac{1}{2},+})$ and $\bar{A}_j^{n,-} \in \mathcal{I}(A_{j-\frac{1}{2}}^{n,-}, A_{j+\frac{1}{2}}^{n,-})$, for $j \in \mathbb{Z}$.

Next, expanding $A_{j+\frac{3}{2}}^{n,-} - 2A_{j+\frac{1}{2}}^{n,-} + A_{j-\frac{1}{2}}^{n,-} = \left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right) + \frac{1}{2}\left(s_{j+1}^n - 2s_j^n + s_{j-1}^n\right)$, and using the definition of the discrete convolution from (3.2.8) together with Theorems 3.3.5 and 3.3.6, we obtain

$$
\begin{aligned}
|A_{j+1}^n - 2A_j^n + A_{j-1}^n| &= |\Delta x\sum_{j\in\mathbb{Z}}(\mu_{j+1-l} - 2\mu_{j-l} + \mu_{j-1-l})\rho_l^n|\\
&= \Delta x^3|\sum_{j\in\mathbb{Z}}\mu''(\bar{x}_{j-l})\rho_l^n| \le \Delta x^2\|\mu''\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})},
\end{aligned}
\tag{B.1.10}
$$

for some $\bar{x}_{j-l} \in (x_{j-l-1}, x_{j-l+1})$. Further, noticing that

$$
\begin{aligned}
&\frac{1}{2}\left(s_{j+1}^n - 2s_j^n + s_{j-1}^n\right)\\
&\quad= \frac{1}{2}\theta\left[(A_{j+2}^n - A_{j+1}^n) - (A_{j+1}^n - A_j^n) - (A_j^n - A_{j-1}^n) + (A_{j-1}^n - A_{j-2}^n)\right]\\
&\quad= \frac{1}{2}\theta\left[(A_{j+2}^n - 2A_{j+1}^n + A_j^n) - (A_j^n - 2A_{j-1}^n + A_{j-2}^n)\right],
\end{aligned}
$$

yields

$$\frac{1}{2}|s_{j+1}^n - 2s_j^n + s_{j-1}^n| \leq \theta \Delta x^2 \|\mu''\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}. \tag{B.1.11}$$

In view of (B.1.11) and (B.1.10), we have the estimate

$$|A_{j+\frac{3}{2}}^{n,-} - 2A_{j+\frac{1}{2}}^{n,-} + A_{j-\frac{1}{2}}^{n,-}| \leq (1+\theta)\Delta x^2 \|\mu''\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}. \tag{B.1.12}$$

Now, applying (B.1.12) together with Lemma 3.3.2 and hypotheses (**H1**) and (**H2**), we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\mathcal{L}_j^a| &\leq \frac{1}{2}\left(\alpha + \lambda\|\partial_\rho f\|\right) \frac{\lambda}{2} M(1+\theta)\Delta x^2 \|\mu''\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \sum_{j\in\mathbb{Z}} |\rho_{j+\frac{1}{2}}^{n,-}| \\
&\leq \frac{1}{2}\left(\alpha + \lambda\|\partial_\rho f\|\right) \frac{\lambda}{2} M(1+\theta)^2 \Delta x \|\mu''\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \Delta x \sum_{j\in\mathbb{Z}} \rho_j^n \\
&\leq \Delta t \frac{1}{4}\left(\alpha + \lambda\|\partial_\rho f\|\right) M(1+\theta)^2 \|\mu''\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}^2.
\end{aligned} \tag{B.1.13}$$

Moving to the estimation of the sum $\sum_{j\in\mathbb{Z}} |\mathcal{L}_j^b|$, we bound the difference $\bar{A}_j^{n,-} - \bar{A}_{j+1}^{n,-}$ using the estimate (3.3.40) as

$$\begin{aligned}
|\bar{A}_j^{n,-} - \bar{A}_{j+1}^{n,-}| &= |\gamma_1 A_{j-\frac{1}{2}}^{n,-} + (1-\gamma_1) A_{j+\frac{1}{2}}^{n,-} - (1-\gamma_2) A_{j+\frac{1}{2}}^{n,-} - \gamma_2 A_{j+\frac{3}{2}}^{n,-}| \\
&\leq \gamma_1 |A_{j-\frac{1}{2}}^{n,-} - A_{j+\frac{1}{2}}^{n,-}| + \gamma_2 |A_{j+\frac{1}{2}}^{n,-} - A_{j+\frac{3}{2}}^{n,-}| \\
&\leq 2(1+\theta)\Delta x \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})},
\end{aligned} \tag{B.1.14}$$

for some $\gamma_1, \gamma_2 \in (0,1)$. Now, invoking (3.3.40), hypothesis (**H2**), property (3.3.9), (B.1.14) and Lemma 3.3.2, we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\mathcal{L}_j^b| &\leq \Delta x K \lambda \left(\|\partial_{\rho A}^2 f\| \sum_{j\in\mathbb{Z}} |\rho_{j-\frac{1}{2}}^{n,-} - \rho_{j+\frac{1}{2}}^{n,-}| + M \sum_{j\in\mathbb{Z}} |\bar{A}_j^{n,-} - \bar{A}_{j+1}^{n,-}| |\rho_{j+\frac{1}{2}}^{n,-}| \right) \\
&\leq \Delta t K \left(\|\partial_{\rho A}^2 f\| \sum_{j\in\mathbb{Z}} |\rho_j^n - \rho_{j-1}^n| + 2M(1+\theta)^2 \|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}^2 \right),
\end{aligned} \tag{B.1.15}$$

where $K := (1+\theta)\|\mu'\| \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})} \frac{1}{4}(\alpha + \lambda\|\partial_\rho f\|)$. Next, aiming to estimate the sum $\sum_{j\in\mathbb{Z}} |\mathcal{L}_j^c|$, we first consider the difference $\bar{\rho}_{j-1}^{n+\frac{1}{2},+} - \bar{\rho}_j^{n+\frac{1}{2},+}$ and write

$$\begin{aligned}
\bar{\rho}_j^{n+\frac{1}{2},+} - \bar{\rho}_{j-1}^{n+\frac{1}{2},+} &= \tilde{\gamma}_1 \rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+} + (1-\tilde{\gamma}_1) \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+} - \tilde{\gamma}_2 \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+} - (1-\tilde{\gamma}_2) \rho_{j-\frac{3}{2}}^{n+\frac{1}{2},+} \\
&= \tilde{\gamma}_1 (\rho_{j+\frac{1}{2}}^{n+\frac{1}{2},+} - \rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+}) + (1-\tilde{\gamma}_2)(\rho_{j-\frac{1}{2}}^{n+\frac{1}{2},+} - \rho_{j-\frac{3}{2}}^{n+\frac{1}{2},+}),
\end{aligned} \tag{B.1.16}$$

for some $\tilde{\gamma}_1, \tilde{\gamma}_2 \in (0,1)$. Now, proceeding analogously to (3.3.56), we write

$$
\begin{aligned}
\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}} = &\left(\rho^{n,-}_{j+\frac{3}{2}} - \rho^{n,-}_{j+\frac{1}{2}}\right)\left(-\frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,-}_{j+1}, A^{n,-}_{j+\frac{3}{2}})\right) \\
&+ \left(\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}\right)\left(1 + \frac{\lambda}{2}\partial_\rho f(\bar{\rho}^{n,+}_{j}, A^{n,+}_{j+\frac{1}{2}})\right) \\
&- \frac{\lambda}{2}\partial_A f(\rho^{n,-}_{j+\frac{1}{2}}, \bar{A}^{n,-}_{j+1})\left(A^{n,-}_{j+\frac{3}{2}} - A^{n,-}_{j+\frac{1}{2}}\right) \\
&+ \frac{\lambda}{2}\partial_A f(\rho^{n,+}_{j-\frac{1}{2}}, \bar{A}^{n,+}_{j})\left(A^{n,+}_{j+\frac{1}{2}} - A^{n,+}_{j-\frac{1}{2}}\right),
\end{aligned}
\tag{B.1.17}
$$

and subsequently apply hypothesis (**H2**) and the estimate (3.3.40), to yield

$$
\begin{aligned}
|\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| \leq &\frac{\lambda}{2}\|\partial_\rho f\||\rho^{n,-}_{j+\frac{3}{2}} - \rho^{n,-}_{j+\frac{1}{2}}| + |\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}|\left(1 + \frac{\lambda}{2}\|\partial_\rho f\|\right) \\
&+ \frac{\lambda}{2}M(|\rho^{n,-}_{j+\frac{1}{2}}| + |\rho^{n,+}_{j-\frac{1}{2}}|)(1+\theta)\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}.
\end{aligned}
\tag{B.1.18}
$$

In (B.1.16), taking the absolute values and summing over $j \in \mathbb{Z}$, and subsequently invoking the estimate (B.1.18), Theorems 3.3.5 and 3.3.6, hypothesis (**H2**), Lemma 3.3.2 and property (3.3.9), it follows that

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}|\bar{\rho}^{n+\frac{1}{2},+}_{j-1} - \bar{\rho}^{n+\frac{1}{2},+}_{j}| \\
&\quad\leq \sum_{j\in\mathbb{Z}}|\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| + \sum_{j\in\mathbb{Z}}|\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{3}{2}}| \\
&\quad\leq \lambda\|\partial_\rho f\|\sum_{j\in\mathbb{Z}}|\rho^{n,-}_{j+\frac{1}{2}} - \rho^{n,-}_{j-\frac{1}{2}}| + (2 + \lambda\|\partial_\rho f\|)\sum_{j\in\mathbb{Z}}|\rho^{n,+}_{j+\frac{1}{2}} - \rho^{n,+}_{j-\frac{1}{2}}| \\
&\qquad+ 2\lambda M(1+\theta)^2\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\Delta x\sum_{j\in\mathbb{Z}}\rho^n_j \\
&\quad\leq 2\left(1 + \lambda\|\partial_\rho f\|\right)\sum_{j\in\mathbb{Z}}|\rho^n_j - \rho^n_{j-1}| + 2\lambda M(1+\theta)^2\|\mu'\|\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})},
\end{aligned}
\tag{B.1.19}
$$

Finally, in view of the estimates (3.3.40) and (B.1.19), we arrive at

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}|\mathcal{L}^c_j| \leq &\frac{\lambda^2}{4}(1+\theta)\Delta x\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\|\partial_A f\|\|\partial^2_{\rho\rho}f\|\sum_{j\in\mathbb{Z}}|\bar{\rho}^{n+\frac{1}{2},+}_{j-1} - \bar{\rho}^{n+\frac{1}{2},+}_{j}| \\
\leq &\frac{\lambda}{2}\Delta t(1+\theta)\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\|\partial_A f\|\|\partial^2_{\rho\rho}f\|\left(1 + \lambda\|\partial_\rho f\|\right)\sum_{j\in\mathbb{Z}}|\rho^n_j - \rho^n_{j-1}| \\
&+ \frac{\lambda^2}{2}\Delta t M(1+\theta)^3\|\mu'\|^2\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}\|\partial_A f\|\|\partial^2_{\rho\rho}f\|.
\end{aligned}
\tag{B.1.20}
$$

Next, applying the estimates (3.3.40) and (3.3.47), we obtain a bound on the sum $\sum_{j\in\mathbb{Z}}|\mathcal{L}^d_j|$ as follows:

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}|\mathcal{L}^d_j| &\leq \frac{1}{4}\Delta t^2(1+\theta)^2\|\mu'\|^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\|\partial_A f\|\|\partial^2_{\rho A}f\|(1 + \lambda\|\partial_\rho f\|) \\
&\leq \frac{1}{4}\Delta t(1+\theta)^2\|\mu'\|^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\|\partial_A f\|\|\partial^2_{\rho A}f\|(1 + \lambda\|\partial_\rho f\|),
\end{aligned}
\tag{B.1.21}
$$

where the last inequality holds for $\Delta t \leq 1$. Collecting the estimates (B.1.13), (B.1.15), (B.1.20) and (B.1.21) we arrive at

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\mathcal{L}_j^1| &\leq \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^a| + \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^b| + \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^c| + \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^d| \\ &\leq \mathcal{K}_1\Delta t + \mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}}|\rho_j^n - \rho_{j-1}^n|,\end{aligned} \tag{B.1.22}$$

where

$$\begin{aligned}\mathcal{K}_1 &= (1+\theta)^2\|\mu''\|\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\frac{1}{4}\left(\alpha+\lambda\|\partial_\rho\|\right)M \\ &\quad + \frac{1}{2}(1+\theta)^3\|\mu'\|^2\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}\left(\alpha+\lambda\|\partial_\rho f\|+\lambda^2\|\partial_A f\|\|\partial^2_{\rho\rho}f\|\right)M \\ &\quad + (1+\theta)^2\|\mu'\|^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\frac{1}{4}\|\partial_A f\|\|\partial^2_{\rho A}f\|(1+\lambda\|\partial_\rho f\|), \\ \mathcal{K}_2 &:= (1+\theta)\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\left(\frac{1}{4}(\alpha+\lambda\|\partial_\rho f\|)\|\partial_{\rho A}f\| + \frac{\lambda}{2}\|\partial_A f\|\|\partial^2_{\rho\rho}f\|\left(1+\lambda\|\partial_\rho f\|\right)\right).\end{aligned} \tag{B.1.23}$$

An analogous treatment of the term $\mathcal{L}_j^2$ yields

$$\sum_{j\in\mathbb{Z}}|\mathcal{L}_j^2| \leq \mathcal{K}_1\Delta t + \mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}}|\rho_j^n - \rho_{j-1}^n|. \tag{B.1.24}$$

Now, the estimates (B.1.22) and (B.1.24) on (B.1.4), it follows that

$$\sum_{j\in\mathbb{Z}}|\hat{\ell}^n_{j+\frac{1}{2}} - \hat{\ell}^n_{j-\frac{1}{2}}| \leq \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^1| + \sum_{j\in\mathbb{Z}}|\mathcal{L}_j^2| \leq 2\mathcal{K}_1\Delta t + 2\mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}}|\rho_j^n - \rho_{j-1}^n|. \tag{B.1.25}$$

In a similar way, we obtain a bound

$$\sum_{j\in\mathbb{Z}}|\hat{k}^n_{j+\frac{1}{2}} - \hat{k}^n_{j-\frac{1}{2}}| \leq 2\mathcal{K}_1\Delta t + 2\mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}}|\rho_j^n - \rho_{j-1}^n|. \tag{B.1.26}$$

Thus, in view of (B.1.25) and (B.1.26), we conclude from (3.3.62) that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\hat{C}^n_{j+\frac{1}{2}}| &\leq \sum_{j\in\mathbb{Z}}|\hat{k}^n_{j+\frac{1}{2}} - \hat{k}^n_{j-\frac{1}{2}}| + \sum_{j\in\mathbb{Z}}|\hat{\ell}^n_{j+\frac{1}{2}} - \hat{\ell}^n_{j-\frac{1}{2}}| \\ &\leq 4\mathcal{K}_1\Delta t + 4\mathcal{K}_2\Delta t\sum_{j\in\mathbb{Z}}|\rho_{j+1}^n - \rho_j^n|.\end{aligned} \tag{B.1.27}$$

B.1.3 Bound on $\lambda \sum_{j\in\mathbb{Z}} |D^n_{j+\frac{1}{2}}|$

Plugging in the definition of the numerical flux (3.2.13) in the expression (3.3.53), rearranging the terms and subsequently applying the mean value theorem, we write

$$
\begin{aligned}
D^n_{j+\frac{1}{2}} &= \frac{1}{2}\left(f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) - f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\\
&\quad + \frac{1}{2}\left(f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{3}{2}}) - f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\right)\\
&\quad - \frac{1}{2}\left(f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\\
&\quad - \frac{1}{2}\left(f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\\
&= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1}) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1})\right]\\
&\quad - \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j}) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j})\right]
\end{aligned}
\tag{B.1.28}
$$

for $\bar{A}^{n+\frac{1}{2}}_j, \tilde{A}^{n+\frac{1}{2}}_j \in \mathcal{I}(A^{n+\frac{1}{2}}_{j-\frac{1}{2}}, A^{n+\frac{1}{2}}_{j+\frac{1}{2}}), j \in \mathbb{Z}$.

Next, we add and subtract the term

$$
\frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1}) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1})\right]
$$

to (B.1.28) and write

$$
\begin{aligned}
\mathcal{D}_{j+\frac{1}{2}} &= \frac{1}{2}\left((A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}}) - (A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\right)\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1}) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1})\right]\\
&\quad + \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1}) - \mathcal{S}_1 + \mathcal{S}_1 - \partial_A f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j})\right]\\
&\quad + \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1}) - \mathcal{S}_2 + \mathcal{S}_2 - \partial_A f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j})\right],
\end{aligned}
\tag{B.1.29}
$$

where $\mathcal{S}_1 := \partial_A f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1})$ and $\mathcal{S}_2 := \partial_A f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1})$. Once again applying the mean value theorem in (B.1.29) yields

$$
\mathcal{D}_{j+\frac{1}{2}} = \mathcal{D}^a_{j+\frac{1}{2}} + \mathcal{D}^b_{j+\frac{1}{2}} + \mathcal{D}^c_{j+\frac{1}{2}} + \mathcal{D}^d_{j+\frac{1}{2}} + \mathcal{D}^e_{j+\frac{1}{2}},
\tag{B.1.30}
$$

where

$$
\begin{aligned}
\mathcal{D}^a_{j+\frac{1}{2}} &:= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - 2A^{n+\frac{1}{2}}_{j+\frac{1}{2}} + A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial_A f(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}, \bar{A}^{n+\frac{1}{2}}_{j+1}) + \partial_A f(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}, \tilde{A}^{n+\frac{1}{2}}_{j+1})\right],\\
\mathcal{D}^b_{j+\frac{1}{2}} &:= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial^2_{\rho A} f(\bar{\rho}^{n+\frac{1}{2},-}_{j}, \bar{A}^{n+\frac{1}{2}}_{j+1})(\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}})\right],\\
\mathcal{D}^c_{j+\frac{1}{2}} &:= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial^2_{AA} f(\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}, \bar{\bar{A}}^{n+\frac{1}{2}}_{j+\frac{1}{2}})(\bar{A}^{n+\frac{1}{2}}_{j+1} - \bar{A}^{n+\frac{1}{2}}_{j})\right],\\
\mathcal{D}^d_{j+\frac{1}{2}} &:= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial^2_{\rho A} f(\bar{\rho}^{n+\frac{1}{2},+}_{j}, \tilde{A}^{n+\frac{1}{2}}_{j+1})(\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}})\right],\\
\mathcal{D}^e_{j+\frac{1}{2}} &:= \frac{1}{2}(A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}})\left[\partial^2_{AA} f(\rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}, \tilde{\tilde{A}}^{n+\frac{1}{2}}_{j+\frac{1}{2}})(\tilde{A}^{n+\frac{1}{2}}_{j+1} - \tilde{A}^{n+\frac{1}{2}}_{j})\right],
\end{aligned}
\tag{B.1.31}
$$

for some $\bar{\bar{A}}^{n+\frac{1}{2}}_{j+\frac{1}{2}} \in \mathcal{I}(\bar{A}^{n+\frac{1}{2}}_{j}, \bar{A}^{n+\frac{1}{2}}_{j+1})$ and $\bar{\rho}^{n+\frac{1}{2},\pm}_{j} \in \mathcal{I}(\rho^{n+\frac{1}{2},\pm}_{j-\frac{1}{2}}, \rho^{n+\frac{1}{2},\pm}_{j+\frac{1}{2}}), j \in \mathbb{Z}$.

Next our goal is to estimate the sum over $j \in \mathbb{Z}$ of absolute values of all the terms in (B.1.28). To estimate the term $\sum_{j\in\mathbb{Z}}|\mathcal{D}^a_{j+\frac{1}{2}}|$, we first expand $A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - 2A^{n+\frac{1}{2}}_{j+\frac{1}{2}} + A^{n+\frac{1}{2}}_{j-\frac{1}{2}}$, sum its absolute value over $j \in \mathbb{Z}$ and successively apply the mean value theorem twice to write

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - 2A^{n+\frac{1}{2}}_{j+\frac{1}{2}} + A^{n+\frac{1}{2}}_{j-\frac{1}{2}}| &\leq \frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}|\mu_{j+2-l} - 2\mu_{j+1-l} + \mu_{j-l}||\rho^{n+\frac{1}{2},+}_{l-\frac{1}{2}}| \\ &\quad + \frac{\Delta x}{2}\sum_{j\in\mathbb{Z}}|\mu_{j+1-l} - 2\mu_{j-l} + \mu_{j-1-l}||\rho^{n+\frac{1}{2},-}_{l+\frac{1}{2}}| \\ &\leq \Delta x^2\|\mu''\|(1+\theta)(1+\lambda\|\partial_\rho f\|)\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})},\end{aligned} \tag{B.1.32}$$

where the last inequality follows from the estimate (3.3.11). Further, invoking (B.1.32), hypothesis (**H2**) and (3.3.11), we obtain

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}|\mathcal{D}^a_{j+\frac{1}{2}}| \\ &\quad\leq \frac{1}{2}\Delta x^2\|\mu''\|(1+\theta)(1+\lambda\|\partial_\rho f\|)\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}M\sum_{j\in\mathbb{Z}}(|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}}| + |\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}}|) \\ &\quad\leq \Delta x^2\|\mu''\|(1+\theta)(1+\lambda\|\partial_\rho f\|)\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}M(1+\theta)\,(1+\lambda\|\partial_\rho f\|)\sum_{j\in\mathbb{Z}}\rho^n_j \\ &\quad\leq \Delta x^2\|\mu''\|(1+\theta)^2(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}M.\end{aligned} \tag{B.1.33}$$

Next, to derive a bound on the term $\sum_{j\in\mathbb{Z}}|\mathcal{D}^c_{j+\frac{1}{2}}|$, we use the estimate (3.3.47) to write

$$\begin{aligned}&|\tilde{A}^{n+\frac{1}{2}}_{j+1} - \tilde{A}^{n+\frac{1}{2}}_{j}|, |\bar{A}^{n+\frac{1}{2}}_{j+1} - \bar{A}^{n+\frac{1}{2}}_{j}| \\ &\quad\leq |\gamma_1 A^{n+\frac{1}{2}}_{j+\frac{3}{2}} + (1-\gamma_1)A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - \gamma_2 A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (1-\gamma_2)A^{n+\frac{1}{2}}_{j-\frac{1}{2}}| \\ &\quad\leq \gamma_1|A^{n+\frac{1}{2}}_{j+\frac{3}{2}} - A^{n+\frac{1}{2}}_{j+\frac{1}{2}}| + (1-\gamma_2)|A^{n+\frac{1}{2}}_{j+\frac{1}{2}} - A^{n+\frac{1}{2}}_{j-\frac{1}{2}}| \\ &\quad\leq 2\Delta x\|\mu'\|(1+\theta)(1+\lambda\|\partial_\rho f\|)\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})},\end{aligned} \tag{B.1.34}$$

for some $\gamma_1, \gamma_2 \in (0,1)$. Now, invoking the estimates (3.3.47) and (B.1.34), hypothesis (**H2**) and Lemma 3.3.4, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\mathcal{D}^c_{j+\frac{1}{2}}| &\leq \Delta x^2 M\|\mu'\|^2(1+\theta)^2(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\sum_{j\in\mathbb{Z}}|\rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}| \\ &\leq \Delta x M\|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)^3\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}\Delta x\sum_{j\in\mathbb{Z}}\rho^n_{j-1} \\ &\leq \Delta x M\|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)^3\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})},\end{aligned} \tag{B.1.35}$$

where the last inequality holds for $\Delta x \leq 1$. Analogously, we can estimate the term $\sum_{j\in\mathbb{Z}}|\mathcal{D}^e_{j+\frac{1}{2}}|$ as follows

$$\sum_{j\in\mathbb{Z}}|\mathcal{D}^e_{j+\frac{1}{2}}| \leq \Delta x M\|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)^3\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}. \tag{B.1.36}$$

Next, we proceed to estimate the terms $\sum_{j\in\mathbb{Z}}|\mathcal{D}^b_{j+\frac{1}{2}}|$ and $\sum_{j\in\mathbb{Z}}|\mathcal{D}^d_{j+\frac{1}{2}}|$. To this end, we sum (B.1.18) over $j \in \mathbb{Z}$ and subsequently apply property (3.3.9) and Lemma 3.3.2 to yield

$$\begin{aligned}\sum_{j\in\mathbb{Z}}&|\rho^{n+\frac{1}{2},+}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},+}_{j-\frac{1}{2}}| \\ &\leq (1+\lambda\|\partial_\rho f\|)\sum_{j\in\mathbb{Z}}|\rho^n_{j+1}-\rho^n_j| + \lambda(1+\theta)^2\|\mu'\|\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}M\Delta x\sum_{j\in\mathbb{Z}}\rho^n_j \\ &\leq (1+\lambda\|\partial_\rho f\|)\sum_{j\in\mathbb{Z}}|\rho^n_{j+1}-\rho^n_j| + \lambda(1+\theta)^2\|\mu'\|\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}M.\end{aligned} \tag{B.1.37}$$

Similarly, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}&|\rho^{n+\frac{1}{2},-}_{j+\frac{1}{2}} - \rho^{n+\frac{1}{2},-}_{j-\frac{1}{2}}| \\ &\leq (1+\lambda\|\partial_\rho f\|)\sum_{j\in\mathbb{Z}}|\rho^n_{j+1}-\rho^n_j| + \lambda(1+\theta)^2\|\mu'\|\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}M.\end{aligned} \tag{B.1.38}$$

Now, invoking the estimates (B.1.37), (B.1.38) and (3.3.47), we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}&|\mathcal{D}^b_{j+\frac{1}{2}}|, \sum_{j\in\mathbb{Z}}|\mathcal{D}^d_{j+\frac{1}{2}}| \\ &\leq \frac{1}{2}\Delta x\|\mu'\|(1+\theta)(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\|\partial^2_{\rho A}f\|\sum_{j\in\mathbb{Z}}|\rho^n_{j+1}-\rho^n_j| \\ &\quad + \Delta t\frac{1}{2}\|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}\|\partial^2_{\rho A}f\|M.\end{aligned} \tag{B.1.39}$$

Finally, the estimates (B.1.33), (B.1.39), (B.1.35) and (B.1.36) together yield the desired estimate:

$$\lambda\sum_{j\in\mathbb{Z}}|D^n_{j+\frac{1}{2}}| \leq \Delta t\mathcal{K}_7 + \Delta t\mathcal{K}_8\sum_{j\in\mathbb{Z}}|\rho^n_{j+1}-\rho^n_j|, \tag{B.1.40}$$

where

$$\begin{aligned}\mathcal{K}_7 &:= \|\mu''\|(1+\theta)^2(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R})}M \\ &\quad + \|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}\|\partial^2_{\rho A}f\|M \\ &\quad + 2\Delta x\|\mu'\|^2(1+\theta)^3(1+\lambda\|\partial_\rho f\|)^3\|\rho_0\|^3_{\mathrm{L}^1(\mathbb{R})}M, \\ \mathcal{K}_8 &:= \|\mu'\|(1+\theta)(1+\lambda\|\partial_\rho f\|)^2\|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}\|\partial^2_{\rho A}f\|.\end{aligned} \tag{B.1.41}$$

B.2 A second-order MUSCL scheme with Runge-Kutta time stepping

We combine the MUSCL-type reconstruction (3.2.1) and a Runge-Kutta time-stepping (see [103, 151]) to obtain a two-stage second-order method to approximate (5.0.1). Given the cell-average solutions $\{\rho^n_j\}_{j\in\mathbb{Z}}$ at t^n, we apply two consecutive Euler-forward stages to obtain a fully discrete second-order scheme as follows:

Step 1. Define

$$\rho_j^{(1)} = \rho_j^n - \lambda[F(\rho^{n,-}_{j+\frac{1}{2}}, \rho^{n,+}_{j+\frac{1}{2}}, A^n_{j+\frac{1}{2}}) - F(\rho^{n,-}_{j-\frac{1}{2}}, \rho^{n,+}_{j-\frac{1}{2}}, A^n_{j-\frac{1}{2}})] \quad \text{for each } \; j \in \mathbb{Z},$$

where F is given by (3.2.15), $\rho^{n,\pm}_{j\mp\frac{1}{2}}, j \in \mathbb{Z}$, are obtained from (3.2.3) and the discrete convolutions are computed using the trapezoidal quadrature rule as follows:

$$A^n_{j+\frac{1}{2}} := \frac{\Delta x}{2} \sum_{l\in\mathbb{Z}} \left[\mu_{j+1-l}\; \rho^{n,+}_{l-\frac{1}{2}} + \mu_{j-l}\; \rho^{n,-}_{l+\frac{1}{2}}\right], \quad j \in \mathbb{Z}.$$

Step 2. Next, we apply the slope limiter (3.2.2) to $\{\rho_j^{(1)}\}_{j\in\mathbb{Z}}$ to obtain the face values $\rho^{(1),\pm}_{j+\frac{1}{2}}, j \in \mathbb{Z}$. Now, define

$$\rho_j^{(2)} = \rho_j^{(1)} - \lambda[F(\rho^{(1),-}_{j+\frac{1}{2}}, \rho^{(1),+}_{j+\frac{1}{2}}, A^{(1)}_{j+\frac{1}{2}}) - F(\rho^{(1),-}_{j-\frac{1}{2}}, \rho^{(1),+}_{j-\frac{1}{2}}, A^{(1)}_{j-\frac{1}{2}})]$$

where

$$A^{(1)}_{j+\frac{1}{2}} := \frac{\Delta x}{2} \sum_{l\in\mathbb{Z}} \left[\mu_{j+1-l}\; \rho^{(1),+}_{l-\frac{1}{2}} + \mu_{j-l}\; \rho^{(1),-}_{l+\frac{1}{2}}\right], \quad j \in \mathbb{Z}.$$

Step 3. Finally, the updated solution at the time level t^{n+1} is computed as

$$\rho_j^{n+1} = \frac{\rho_j^n + \rho_j^{(2)}}{2}, \quad j \in \mathbb{Z}. \tag{B.2.1}$$

Remark B.2.1. The convergence of the scheme (B.2.1) to the entropy solution of problem (5.0.1) can be established using calculations analogous to those for the scheme (3.2.5), within the general framework developed in Chapter 2.

C

Appendix C

C.1 Proof of Lemma 5.5.1

In this section, we provide the proof for the one-sided estimate stated in Lemma 5.5.1. The corresponding estimate for the case when $k \equiv 1$, was derived in [143]. We follow a similar approach for the problem (5.0.1), where the coefficient k can be discontinuous. The proof requires several technical lemmas, which we present in Section C.1.1. We then conclude with the proof of Lemma 5.5.1 in Section C.1.2.

C.1.1 A bound on non-decreasing sequences

In this section, we establish a result regarding the jumps generated by applying the scheme (5.2.4) to non-decreasing sequences with finitely many non-zero jumps. The main result is stated in the following lemma, and its proof is provided at the end of this section.

Lemma C.1.1. *Consider a non-decreasing sequence $\{u_j\}_{j\in\mathbb{Z}}$ such that for some $l, r \in \mathbb{Z}$, the jumps $\delta_{j+\frac{1}{2}} := u_{j+1} - u_j = 0$ for all $j \notin \{l-1, l, \dots, r-1\}$. Further, let $|u_j| \leq \|C_{u_0}\|$, $\forall j \in \mathbb{Z}$, where C_{u_0} is as in* (5.4.3). *Let $\{u'_{j+\frac{1}{2}}\}_{j\in\mathbb{Z}}$ be obtained from $\{u_j\}_{j\in\mathbb{Z}}$ by applying the time-update formula* (5.2.4) *and denote the corresponding jumps $\delta'_j := u'_{j+\frac{1}{2}} - u'_{j-\frac{1}{2}}$ for $j \in \mathbb{Z}$. Under the CFL condition* (5.5.1) *there exists a constant $\Theta \geq 0$ independent of Δx such* *that the jump sequence* $\{\delta'_j\}_{j\in\mathbb{Z}}$ *satisfies the following estimate*

$$\begin{aligned}\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta'_{j-1})^2\right)\geq \frac{1}{500}\lambda\gamma_1\sum_{j=l-1}^{r+1}(\delta_{j-\frac{1}{2}})^3+\frac{1}{6400}\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2\\ -\Theta\sum_{j=l-2}^{r+2}|\Delta k_{j-\frac{3}{2}}|,\end{aligned}\tag{C.1.1}$$

where $\Delta^2\delta_{j-\frac{3}{2}} := \Delta\delta_{j-1}-\Delta\delta_{j-2}$.

To prove Lemma C.1.1, we need to establish several auxiliary lemmas, starting with the following one.

Lemma C.1.2. *Let* $\{u_j\}_{j\in\mathbb{Z}}$ *and* $\{\delta'_j\}_{j\in\mathbb{Z}}$ *be as in Lemma C.1.1. Under the CFL condition* (5.5.1) *the jumps* $\{\delta'_j\}_{j\in\mathbb{Z}}$ *satisfy the following estimate*

$$|\delta'_j|\leq(\delta_{j+\frac{1}{2}}+\delta_{j-\frac{1}{2}})+\lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}|+|\Delta k_{j-\frac{1}{2}}|\right).\tag{C.1.2}$$

Proof. From the scheme (5.2.4), the jump δ'_j can be written as

$$\begin{aligned}\delta'_j=&\frac{1}{2}(\delta_{j+\frac{1}{2}}+\delta_{j-\frac{1}{2}})-\lambda\left(f(k_{j+1},u^{\frac{1}{2}}_{j+1})-2f(k_j,u^{\frac{1}{2}}_j)+f(k_{j-1},u^{\frac{1}{2}}_{j-1})\right)\\ &-\frac{1}{8}(s_{j+1}-2s_j+s_{j-1}),\end{aligned}\tag{C.1.3}$$

where we define $u^{\frac{1}{2}}_j := u_j-\frac{\lambda}{2}f_u(k_j,u_j)s_j$, $s_j := \min\{\delta_{j-\frac{1}{2}},\delta_{j+\frac{1}{2}}\}$, $j\in\mathbb{Z}$. We can also write

$$u^{\frac{1}{2}}_{j+1}-u^{\frac{1}{2}}_j=\delta_{j+\frac{1}{2}}-\frac{\lambda}{2}(a_{j+1}s_{j+1}-a_js_j),$$

where

$$a_j := f_u(k_j,u_j),\ \text{for}\, j\in\mathbb{Z}.\tag{C.1.4}$$

Now, by adding and subtracting the term $f(k_j,u^{\frac{1}{2}}_{j+1})$ and using the mean value theorem, we can write

$$\begin{aligned}f(k_{j+1},u^{\frac{1}{2}}_{j+1})-f(k_j,u^{\frac{1}{2}}_j)&=f_k(c_{12},u^{\frac{1}{2}}_{j+1})\Delta k_{j+\frac{1}{2}}+f_u(k_j,\zeta_{12})(u^{\frac{1}{2}}_{j+1}-u^{\frac{1}{2}}_j)\\ &=\bar{b}_{j+\frac{1}{2}}\Delta k_{j+\frac{1}{2}}+\bar{a}_{j+\frac{1}{2}}\left(\delta_{j+\frac{1}{2}}-\frac{\lambda}{2}(a_{j+1}s_{j+1}-a_js_j)\right),\end{aligned}$$

where we define

$$\bar{a}_{j+\frac{1}{2}} := f_u(k_j,\zeta),\quad \bar{b}_{j+\frac{1}{2}} := f_k(c,u^{\frac{1}{2}}_{j+1}),\tag{C.1.5}$$

for some $\zeta\in\mathcal{I}(u^{\frac{1}{2}}_j,u^{\frac{1}{2}}_{j+1})$ and $c\in\mathcal{I}(k_j,k_{j+1})$. Now, (C.1.3) reduces to

$$\begin{aligned}\delta'_j=&\frac{1}{2}(\delta_{j+\frac{1}{2}}+\delta_{j-\frac{1}{2}})-\frac{1}{8}(\Delta s_{j+\frac{1}{2}}-\Delta s_{j-\frac{1}{2}})-\lambda\left[\bar{b}_{j+\frac{1}{2}}\Delta k_{j+\frac{1}{2}}-\bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}\right]\\ &-\lambda\left[\bar{a}_{j+\frac{1}{2}}\left(\delta_{j+\frac{1}{2}}-\frac{\lambda}{2}(a_{j+1}s_{j+1}-a_js_j)\right)-\bar{a}_{j-\frac{1}{2}}\left(\delta_{j-\frac{1}{2}}-\frac{\lambda}{2}(a_js_j-a_{j-1}s_{j-1})\right)\right],\end{aligned}\tag{C.1.6}$$

The CFL condition (5.5.1) implies that $\kappa \leq \frac{1}{4}$, which yields the following upper bound

$$\begin{aligned} \delta'_j &\leq \left(\frac{1}{2} + \frac{1}{8} + \kappa + \kappa^2\right)(\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) + \lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right) \\ &\leq (\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) + \lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right). \end{aligned} \tag{C.1.7}$$

Analogously, we obtain a lower bound

$$\begin{aligned} \delta'_j &\geq \left(\frac{1}{2} - \frac{1}{8} - \kappa - \kappa^2\right)(\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) - \lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right) \\ &\geq -\lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right). \end{aligned} \tag{C.1.8}$$

The estimates (C.1.7) and (C.1.8) together imply that

$$|\delta'_j| \leq (\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) + \lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right). \tag{C.1.9}$$

This completes the proof. □

Lemma C.1.3. *Let* $\{u_j\}_{j\in\mathbb{Z}}$ *and* $\{\delta'_j\}_{j\in\mathbb{Z}}$ *be as in Lemma C.1.1, with* $|u_j| \leq C_{u_0}$*, for all* $j \in \mathbb{Z}$*, (where* C_{u_0} *is as in* (5.4.3)*). Define* δ''_j *by replacing the term* $(a^n_{j+1}s^n_{j+1} - a^n_j s^n_j)$ *with* $\bar{a}^n_{j+\frac{1}{2}}\Delta s_{j+\frac{1}{2}}$ *in* δ'_j (C.1.6)*, where* $\bar{a}^n_{j+\frac{1}{2}}$ *is as defined in* (C.1.5)*. Then*

$$\left|\sum_{j=l-2}^{r}(\delta'_j)^2 - \sum_{j=l-2}^{r}(\delta''_j)^2\right| \leq 55\lambda^2\gamma_2\|f_u\|\sum_{j=l-2}^{r}(\delta^n_{j+\frac{1}{2}})^3 + P_1, \tag{C.1.10}$$

where $P_1 := (16\lambda^2 C^2_{u_0}\|f_u\|\|f_{uk}\| + 44\lambda^3\|f_k\|\gamma_2\|f_u\|C^2_{u_0} + 16\lambda^3\|f_k\|C_{u_0}\|k\|)$
$\sum_{j=l-2}^{r+1}|\Delta k_{j-\frac{1}{2}}|$.

Proof. The jump term δ''_j is given by

$$\begin{aligned} \delta''_j = &\frac{1}{2}(\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) - \frac{1}{8}(\Delta s_{j+\frac{1}{2}} - \Delta s_{j-\frac{1}{2}}) - \lambda\left(\bar{b}_{j+\frac{1}{2}}\Delta k_{j+\frac{1}{2}} - \bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}\right) \\ &- \lambda\left(\bar{a}_{j+\frac{1}{2}}\left(\delta_{j+\frac{1}{2}} - \frac{\lambda}{2}a_{j+\frac{1}{2}}\Delta s_{j+\frac{1}{2}}\right) - \bar{a}_{j-\frac{1}{2}}\left(\delta_{j-\frac{1}{2}} - \frac{\lambda}{2}\bar{a}_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}\right)\right). \end{aligned} \tag{C.1.11}$$

With arguments analogous to those used in (C.1.7), it follows that

$$|\delta''_j| \leq (\delta_{j+\frac{1}{2}} + \delta_{j-\frac{1}{2}}) + \lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right). \tag{C.1.12}$$

Now, we write the difference $\delta'_j - \delta''_j$ as

$$\delta'_j - \delta''_j = \frac{\lambda^2}{2}(\tau_{j+\frac{1}{2}} - \tau_{j-\frac{1}{2}}), \tag{C.1.13}$$

where $\tau_{j+\frac{1}{2}} := \bar{a}_{j+\frac{1}{2}}\left(a_{j+1}s_{j+1} - a_j s_j - \bar{a}_{j+\frac{1}{2}}\Delta s_{j+\frac{1}{2}}\right)$, which can be bounded as

$$|\tau_{j+\frac{1}{2}}| \leq \|f_u\||s_{j+1}(a_{j+1} - \bar{a}_{j+\frac{1}{2}}) - s_j(a_j - \bar{a}_{j+\frac{1}{2}})|. \tag{C.1.14}$$

Further, adding and subtracting the term $f_u(k_j, u_{j+1})$ in $a_{j+1} - \bar{a}_{j+\frac{1}{2}}$ yields

$$a_{j+1} - \bar{a}_{j+\frac{1}{2}} = f_{uk}(c, u_{j+1})\Delta k_{j+\frac{1}{2}} + f_{uu}(k_j, \zeta')\,(u_{j+1} - \zeta)\,, \tag{C.1.15}$$

where ζ is as in (C.1.5), $c \in \mathcal{I}(k_j, k_{j+1})$ and $\zeta' \in \mathcal{I}(\zeta, u_{j+1})$. Now, due to the CFL condition (5.5.1), it follows that

$$\begin{aligned} u^{\frac{1}{2}}_{j+1} &= u_{j+1} - \frac{1}{2}\lambda a_{j+1}s_{j+1} \le u_{j+1} + \frac{\delta_{j+\frac{1}{2}}}{4} =: u^{+}_{j+1} \quad \text{and} \\ u^{\frac{1}{2}}_{j+1} &= u_{j+1} - \frac{1}{2}\lambda a_{j+1}s_{j+1} \ge u_{j+1} - \frac{\delta_{j+\frac{1}{2}}}{4} =: u^{-}_{j+1}, \quad \text{for } j \in \mathbb{Z}. \end{aligned} \tag{C.1.16}$$

The CFL condition (5.5.1) and (C.1.16) ensure that $u^{\frac{1}{2}}_j \in [u^-_j, u_{j+1}]$ and $u^{\frac{1}{2}}_{j+1} \in [u_j, u^+_{j+1}]$. Combining this with the fact that $\zeta \in \mathcal{I}(u^{\frac{1}{2}}_j, u^{\frac{1}{2}}_{j+1})$, we derive the following estimate:

$$|u_{j+1} - \zeta| \le \max\{u_{j+1} - u^-_j, u^+_{j+1} - u_{j+1}\} \le \max\left\{\delta_{j+\frac{1}{2}} + \frac{\delta_{j-\frac{1}{2}}}{4}, \frac{\delta_{j+\frac{1}{2}}}{4}\right\} \le \delta_{j+\frac{1}{2}} + \frac{\delta_{j-\frac{1}{2}}}{4}, \tag{C.1.17}$$

which, when combined with (C.1.15) and hypothesis **H2**, provides the following upper bound

$$|a_{j+1} - \bar{a}_{j+\frac{1}{2}}| \le \|f_{uk}\||\Delta k_{j+\frac{1}{2}}| + \gamma_2\left(\delta_{j+\frac{1}{2}} + \frac{\delta_{j-\frac{1}{2}}}{4}\right). \tag{C.1.18}$$

In a similar way, we have

$$|a_j - \bar{a}_{j+\frac{1}{2}}| = |f_u(k_j, u_j) - f_u(k_j, \zeta)| = |f_{uu}(k_j, \bar{\zeta})\,(u_j - \zeta)| \le \gamma_2|u_j - \zeta|, \tag{C.1.19}$$

where $\bar{\zeta} \in \mathcal{I}(\zeta, u_j)$. Here, analogous to (C.1.17), $|u_j - \zeta|$ can be bounded as

$$|u_j - \zeta| \le \max\{u_j - u^-_j, u^+_{j+1} - u_j\} \le \max\left\{\frac{\delta_{j-\frac{1}{2}}}{4}, \delta_{j+\frac{1}{2}} + \frac{\delta_{j+\frac{1}{2}}}{4}\right\} \le \frac{5}{4}\delta_{j+\frac{1}{2}} + \frac{1}{4}\delta_{j-\frac{1}{2}}. \tag{C.1.20}$$

Using (C.1.18), (C.1.19) and (C.1.20) in (C.1.14), we obtain an estimate on the term $\tau_{j+\frac{1}{2}}$ in (C.1.13):

$$|\tau_{j+\frac{1}{2}}| \le \gamma_2\|f_u\|\delta_{j+\frac{1}{2}}\left(\frac{9}{4}\delta_{j+\frac{1}{2}} + \frac{1}{2}\delta_{j-\frac{1}{2}}\right) + \|f_u\|\|f_{uk}\|\delta_{j+\frac{1}{2}}|\Delta k_{j+\frac{1}{2}}|. \tag{C.1.21}$$

The estimate (C.1.21) applied on (C.1.13) yields

$$\begin{aligned} |\delta'_j - \delta''_j| \le{} & \frac{\lambda^2}{2}\gamma_2\|f_u\|\left(\frac{9}{4} + \frac{1}{2} + \frac{9}{4} + \frac{1}{2}\right)M_{\delta_{j-\frac{1}{2}}} \\ & + \frac{\lambda^2}{2}\|f_u\|\|f_{uk}\|\left(\delta_{j+\frac{1}{2}}|\Delta k_{j+\frac{1}{2}}| + \delta_{j-\frac{1}{2}}|\Delta k_{j-\frac{1}{2}}|\right), \end{aligned} \tag{C.1.22}$$

where $M_{\delta_{j-\frac{1}{2}}} := \max\left\{(\delta_{j+\frac{1}{2}})^2, (\delta_{j-\frac{1}{2}})^2, (\delta_{j-\frac{3}{2}})^2\right\}$. Making use of the estimate (C.1.22) together with (C.1.9) and (C.1.12) and using the inequality $ab^2 \le a^3 + b^3$, for $a, b \ge 0$, it follows that

$$\Big|\sum_{j=l-2}^{r}(\delta'_j)^2-\sum_{j=l-2}^{r}(\delta''_j)^2\Big|\le\sum_{j=l-2}^{r}|\delta'_j-\delta''_j||\delta'_j+\delta''_j|\le 55\lambda^2\gamma_2\|f_u\|\sum_{j=l-2}^{r}(\delta_{j+\frac{1}{2}})^3+P_1. \tag{C.1.23}$$

To obtain P_1, we have used the bound $|\delta_{j+\frac{1}{2}}|\le 2C_{u_0}$. This completes the proof. $\square$

Now we define the term $\mathcal{D}:=\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta''_{j-1})^2\right)$ and write this in a suitable form to obtain a lower bound. We begin with the notations

$$\alpha_{j-\frac{1}{2}}:=\frac{1}{2}+\lambda\bar{a}_{j-\frac{1}{2}} \text{ and } \varphi_{j-\frac{1}{2}}:=\alpha_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}}), \tag{C.1.24}$$

and reformulate the modified jumps (C.1.11) as

$$\begin{aligned}\delta''_{j-1}=(1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{1}{2}}+\alpha_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}}-\frac{1}{2}\left(\varphi_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}\Delta s_{j-\frac{3}{2}}\right)\\ -\lambda\left(\bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}-\bar{b}_{j-\frac{3}{2}}\Delta k_{j-\frac{3}{2}}\right).\end{aligned} \tag{C.1.25}$$

Plugging in the expression (C.1.25), we can write $\mathcal{D}$ as

$$\mathcal{D}=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+P_2, \tag{C.1.26}$$

where

$$\mathcal{I}_1:=\sum_{j=l-1}^{r+1}\delta^2_{j-\frac{1}{2}}-\sum_{j=l-1}^{r+1}\left(\alpha_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}}+(1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{1}{2}}\right)^2, \tag{C.1.27}$$

$$\mathcal{I}_2:=\sum_{j=l-1}^{r+1}\left((1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{1}{2}}+\alpha_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}}\right)\left(\varphi_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}\Delta s_{j-\frac{3}{2}}\right),$$

$$\mathcal{I}_3:=-\frac{1}{4}\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}\Delta s_{j-\frac{3}{2}}\right)^2,$$

$$\begin{aligned}P_2:=&-\lambda^2\sum_{j=l-1}^{r+1}\left(\bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}-\bar{b}_{j-\frac{3}{2}}\Delta k_{j-\frac{3}{2}}\right)^2\\ &+2\lambda\sum_{j=l-1}^{r+1}\left(\bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}-\bar{b}_{j-\frac{3}{2}}\Delta k_{j-\frac{3}{2}}\right)\left(\alpha_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}}+(1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{1}{2}}\right)\\ &-\lambda\sum_{j=l-1}^{r+1}\left(\bar{b}_{j-\frac{1}{2}}\Delta k_{j-\frac{1}{2}}-\bar{b}_{j-\frac{3}{2}}\Delta k_{j-\frac{3}{2}}\right)\left(\varphi_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}\Delta s_{j-\frac{3}{2}}\right).\end{aligned}$$

Noting that $|\alpha_{j-\frac{1}{2}}|\le 1$ and $|\varphi_{j-\frac{1}{2}}|\le\frac{1}{2}$, for $j\in\mathbb{Z}$ (by the CFL condition (5.5.1)), we obtain

$$|P_2|\le\lambda\|f_k\|(8\lambda\|f_k\|\|k\|+24C_{u_0}). \tag{C.1.28}$$

Upon a change of index in the summation and grouping the terms appropriately, the term $\mathcal{I}_1$ can be rewritten as follows:

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{j=l-1}^{r+1} \left(\delta^2_{j-\frac{1}{2}}\left(1-\alpha^2_{j-\frac{1}{2}}-(1-\alpha_{j-\frac{1}{2}})^2\right)-2\alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{3}{2}}\delta_{j-\frac{1}{2}}\right) \qquad \text{(C.1.29)}\\
&= \sum_{j=l-1}^{r+1} \left(\alpha_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})\delta^2_{j-\frac{1}{2}}+\alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{3}{2}})\delta^2_{j-\frac{3}{2}}-2\alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{1}{2}})\delta_{j-\frac{3}{2}}\delta_{j-\frac{1}{2}}\right)\\
&= \sum_{j=l-1}^{r+1} \alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{1}{2}})(\delta_{j-\frac{1}{2}}-\delta_{j-\frac{3}{2}})^2\\
&\quad+\sum_{j=l-1}^{r+1}(\alpha_{j-\frac{1}{2}}-\alpha_{j-\frac{3}{2}})\left(\alpha_{j-\frac{3}{2}}\delta^2_{j-\frac{3}{2}}+(1-\alpha_{j-\frac{1}{2}})\delta^2_{j-\frac{1}{2}}\right).
\end{aligned}$$

Now, we write

$$\mathcal{I}_3 = -\frac{1}{4}\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}}\Delta^2 s_{j-1}\right)^2+\mathcal{E}_1, \qquad \text{(C.1.30)}$$

where $\mathcal{E}_1 := \mathcal{I}_3 + \frac{1}{4}\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}}\Delta^2 s_{j-1}\right)^2$. Incorporating the expressions (C.1.29) and (C.1.30) into (C.1.26) and by adding and subtracting $\frac{1}{2}\sum_{j=l-1}^{r+1}(\varphi_{j-\frac{3}{2}}\Delta^2\delta_{j-\frac{3}{2}})$ the term $\mathcal{D}$ is reformulated as

$$\mathcal{D} = \mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3+P_2+\mathcal{E}_1, \qquad \text{(C.1.31)}$$

where

$$\begin{aligned}
\mathcal{Q}_1 &:= \sum_{j=l-1}^{r+1}\alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-\frac{1}{2}})^2+\mathcal{I}_2-\frac{1}{2}\sum_{j=l-1}^{r+1}(\varphi_{j-\frac{3}{2}}\Delta^2\delta_{j-\frac{3}{2}}), \qquad \text{(C.1.32)}\\
\mathcal{Q}_2 &:= \frac{1}{4}\left(2\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}}\Delta^2\delta_{j-\frac{3}{2}}\right)^2-\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}}\Delta^2 s_{j-1}\right)^2\right) \quad \text{and}\\
\mathcal{Q}_3 &:= \sum_{j=l-1}^{r+1}\Delta\alpha_{j-1}\left(\alpha_{j-\frac{3}{2}}(\delta_{j-\frac{3}{2}})^2+(1-\alpha_{j-\frac{1}{2}})(\delta_{j-\frac{1}{2}})^2\right).
\end{aligned}$$

Remark C.1.4. It is possible to derive a bound for the term $\mathcal{E}_1$ in (C.1.31). To show this, we begin by rewriting it as

$$\begin{aligned}
\mathcal{E}_1 = &-\frac{1}{4}\sum_{j=l-1}^{r+1}\left((\Delta s_{j-\frac{1}{2}})^2\left(\varphi_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}\right)\left(\varphi_{j-\frac{1}{2}}+\varphi_{j-\frac{3}{2}}\right)\right)\\
&-\frac{1}{4}\sum_{j=l-1}^{r+1}\left(2\left(\varphi_{j-\frac{3}{2}}-\varphi_{j-\frac{1}{2}}\right)\varphi_{j-\frac{3}{2}}\Delta s_{j-\frac{1}{2}}\Delta s_{j-\frac{3}{2}}\right).
\end{aligned} \qquad \text{(C.1.33)}$$

Immediately, we have the following bound:

$$|\varphi_{j-\frac{1}{2}}-\varphi_{j-\frac{3}{2}}| = \lambda^2|(\bar{a}_{j-\frac{1}{2}})^2-(\bar{a}_{j-\frac{3}{2}})^2| \le \lambda^2|\bar{a}_{j-\frac{1}{2}}-\bar{a}_{j-\frac{3}{2}}||\bar{a}_{j-\frac{1}{2}}+\bar{a}_{j-\frac{3}{2}}|. \qquad \text{(C.1.34)}$$

Next, using the definition of $\bar{a}_{j-\frac{1}{2}}$ from (C.1.5) and adding and subtracting the term $f_u(k_{j-2}, \bar{u}_{j-\frac{1}{2}})$, we get

$$\bar{a}_{j-\frac{1}{2}} - \bar{a}_{j-\frac{3}{2}} = f_{uk}(\bar{k}_{j-\frac{3}{2}}, \bar{u}_{j-\frac{1}{2}})\Delta k_{j-\frac{3}{2}} + f_{uu}(k_{j-2}, \bar{\bar{u}}_{j-1})(\bar{u}_{j-\frac{1}{2}} - \bar{u}_{j-\frac{3}{2}}), \tag{C.1.35}$$

for some $\bar{k}_{j-\frac{3}{2}} \in \mathcal{I}(k_{j-2}, k_{j-1})$, $\bar{u}_{j-\frac{1}{2}} \in \mathcal{I}(u^{\frac{1}{2}}_{j-1}, u^{\frac{1}{2}}_{j})$, $\bar{u}_{j-\frac{3}{2}} \in \mathcal{I}(u^{\frac{1}{2}}_{j-2}, u^{\frac{1}{2}}_{j-1})$ and $\bar{\bar{u}}_{j-1} \in \mathcal{I}(\bar{u}_{j-\frac{3}{2}}, \bar{u}_{j-\frac{1}{2}})$. Further, by the CFL restriction (5.5.1) and using arguments similar to those in (C.1.16), we have $\bar{u}_{j-\frac{1}{2}} \in [u_{j-1} - \frac{\delta_{j-\frac{3}{2}}}{4}, u_j + \frac{\delta_{j-\frac{1}{2}}}{4}]$ and $\bar{u}_{j-\frac{3}{2}} \in [u_{j-2} - \frac{\delta_{j-\frac{3}{2}}}{4}, u_{j-1} + \frac{\delta_{j-\frac{3}{2}}}{4}]$. This in turn, implies that

$$\begin{aligned}
\bar{u}_{j-\frac{1}{2}} - \bar{u}_{j-\frac{3}{2}} &\le u_j + \frac{\delta_{j-\frac{1}{2}}}{4} - u_{j-2} + \frac{\delta_{j-\frac{3}{2}}}{4} \\
&= (u_j - u_{j-1}) + (u_{j-1} - u_{j-2}) + \frac{1}{4}(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}) \le \frac{5}{4}(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}), \\
\bar{u}_{j-\frac{1}{2}} - \bar{u}_{j-\frac{3}{2}} &\ge -\frac{\delta_{j-\frac{3}{2}}}{2},
\end{aligned}$$

and hence

$$|\bar{u}_{j-\frac{1}{2}} - \bar{u}_{j-\frac{3}{2}}| \le \frac{5}{4}(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}), \tag{C.1.36}$$

due to the fact that $\delta_{j-\frac{1}{2}} \ge 0, j \in \mathbb{Z}$. The estimates (C.1.34) and (C.1.36) together with the hypothesis **H2** imply that

$$|\varphi_{j-\frac{1}{2}} - \varphi_{j-\frac{3}{2}}| \le 2\lambda^2 \|f_u\| \gamma_2 \frac{5}{4}(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}) + 2\lambda^2 \|f_u\| \|f_{uk}\| |\Delta k_{j-\frac{3}{2}}|. \tag{C.1.37}$$

By the CFL condition (5.5.1), we have $|\varphi_{j-\frac{1}{2}}| \le \frac{1}{2}, j \in \mathbb{Z}$. Moreover, $|\Delta s_{j-\frac{1}{2}}| \le 2\delta_{j-\frac{1}{2}}, j \in \mathbb{Z}$ and $ab^2 \le a^3 + b^3$, for any $a, b \ge 0$. Invoking these observations and the estimate (C.1.37) into the expression (C.1.33), we obtain the desired bound

$$|\mathcal{E}_1| \le \frac{35}{2}\lambda^2 \|f_u\| \gamma_2 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + 24\lambda^2 \|f_{uk}\| C^2_{u_0} \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}|. \tag{C.1.38}$$

Lemma C.1.5. *Let $\{u_j\}_{j\in\mathbb{Z}}$ be as in Lemma C.1.1. Then for $\alpha_{j-\frac{1}{2}}$ in* (C.1.24),

$$\Delta\alpha_{j-1} \ge \frac{3}{8}\lambda\gamma_1 \left(\delta^n_{j-\frac{1}{2}} + \delta^n_{j-\frac{3}{2}}\right) + \lambda \Delta k_{j-\frac{3}{2}} f_{ku}(\bar{\bar{k}}_{j-\frac{3}{2}}, \bar{u}_{j-\frac{1}{2}}), \tag{C.1.39}$$

for $\bar{\bar{k}}_{j-\frac{3}{2}} \in \mathcal{I}(k_{j-1}, k_{j-2})$.

Proof. Using the fact that $\bar{a}_{j-\frac{1}{2}} = \dfrac{f(k_{j-1}, u^{\frac{1}{2}}_{j}) - f(k_{j-1}, u^{\frac{1}{2}}_{j-1})}{u^{\frac{1}{2}}_{j} - u^{\frac{1}{2}}_{j-1}}$ for $j \in \mathbb{Z}$, the term $\Delta\alpha_{j-1} = \lambda(\bar{a}_{j-\frac{1}{2}} - \bar{a}_{j-\frac{3}{2}})$ can be written as follows

$$\Delta\alpha_{j-1} = \lambda \left(\frac{f(k_{j-2}, u^{\frac{1}{2}}_{j}) - f(k_{j-2}, u^{\frac{1}{2}}_{j-1})}{u^{\frac{1}{2}}_{j} - u^{\frac{1}{2}}_{j-1}} - \frac{f(k_{j-2}, u^{\frac{1}{2}}_{j-1}) - f(k_{j-2}, u^{\frac{1}{2}}_{j-2})}{u^{\frac{1}{2}}_{j-1} - u^{\frac{1}{2}}_{j-2}} \right) \tag{C.1.40}$$

$$+\lambda\left(\frac{\left(f(k_{j-1},u_j^{\frac{1}{2}})-f(k_{j-2},u_j^{\frac{1}{2}})\right)-\left(f(k_{j-1},u_{j-1}^{\frac{1}{2}})-f(k_{j-2},u_{j-1}^{\frac{1}{2}})\right)}{u_j^{\frac{1}{2}}-u_{j-1}^{\frac{1}{2}}}\right)$$

$$=\lambda\bar{f}_{j-2}\left[u_{j-2}^{\frac{1}{2}},u_{j-1}^{\frac{1}{2}},u_j^{\frac{1}{2}}\right]\left(u_j^{\frac{1}{2}}-u_{j-2}^{\frac{1}{2}}\right)+\lambda\Delta k_{j-\frac{3}{2}}\left(\frac{f_k(\bar{k}_{j-\frac{3}{2}},u_j^{\frac{1}{2}})-f_k(\bar{\bar{k}}_{j-\frac{3}{2}},u_{j-1}^{\frac{1}{2}})}{u_j^{\frac{1}{2}}-u_{j-1}^{\frac{1}{2}}}\right)$$

$$=\frac{\lambda}{2}f_{uu}(k_{j-2},\bar{\zeta})\left(u_j^{\frac{1}{2}}-u_{j-2}^{\frac{1}{2}}\right)+\lambda\Delta k_{j-\frac{3}{2}}f_{ku}(\bar{\bar{k}}_{j-\frac{3}{2}},\bar{u}_{j-\frac{1}{2}}),$$

where $f_{j-2}(\cdot)$ denotes the denote the divided difference of the function $f(k_{j-2},\cdot)$, $\bar{\zeta}\in\left(\min\{u_{j-2}^{\frac{1}{2}},u_{j-1}^{\frac{1}{2}},u_j^{\frac{1}{2}}\},\max\{u_{j-2}^{\frac{1}{2}},u_{j-1}^{\frac{1}{2}},u_j^{\frac{1}{2}}\}\right)$, $\bar{u}_{j-\frac{1}{2}}\in\mathcal{I}(u_{j-1}^{\frac{1}{2}},u_j^{\frac{1}{2}})$, and $\bar{k}_{j-\frac{3}{2}},\bar{\bar{k}}_{j-\frac{3}{2}}\in\mathcal{I}(k_{j-1},k_{j-2})$. The second term in the last step of (C.1.40) is derived by applying assumption **H3**, $f_{kk}=0$, and adding and subtracting $f_k(\bar{\bar{k}}_{j-\frac{3}{2}},u_j^{\frac{1}{2}})$.

Now, the CFL condition (5.5.1) allows us to write

$$\begin{aligned}u_j^{\frac{1}{2}}&=u_j^n-\frac{\lambda}{2}f_u(k_j,u_j^n)s_j^n\geq u_j^n-\frac{\delta_{j-\frac{1}{2}}^n}{4}\quad\text{and}\\ u_{j-2}^{\frac{1}{2}}&=u_{j-2}^n-\frac{\lambda}{2}f_u(k_{j-2},u_{j-2}^n)s_{j-2}^n\leq u_{j-2}^n+\frac{\delta_{j-\frac{3}{2}}^n}{4},\end{aligned}\tag{C.1.41}$$

and hence $u_j^{\frac{1}{2}}-u_{j-2}^{\frac{1}{2}}\geq\frac{3}{4}\left(\delta_{j-\frac{1}{2}}^n+\delta_{j-\frac{3}{2}}^n\right)$. Using this bound in (C.1.40) together with **H2**, yields

$$\Delta\alpha_{j-1}\geq\frac{3}{8}\lambda\gamma_1\left(\delta_{j-\frac{1}{2}}^n+\delta_{j-\frac{3}{2}}^n\right)+\lambda\Delta k_{j-\frac{3}{2}}f_{ku}(\bar{\bar{k}}_{j-\frac{3}{2}},\bar{u}_{j-\frac{1}{2}}).\tag{C.1.42}$$

This completes the proof. □

Lemma C.1.6. *Let $\{u_j\}_{j\in\mathbb{Z}}$ be as in Lemma C.1.1. The term $\mathcal{Q}_1$ in* (C.1.32) *can be represented as follows*

$$\mathcal{Q}_1=\mathcal{R}_1+\mathcal{Q}_1^*-\mathcal{E}_4-\mathcal{E}_5-\mathcal{E}_6,\tag{C.1.43}$$

where

$$\begin{aligned}\mathcal{R}_1:=&\sum_{j=l-1}^{r+1}\alpha_{j-\frac{3}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_j)^2+\sum_{j=l-1}^{r+1}\varphi_{j-\frac{1}{2}}\Delta s_{j-\frac{1}{2}}\left(\delta_{j-\frac{1}{2}}(\Delta\alpha_j-\Delta\alpha_{j-1})\right)\\ &-\sum_{\Delta\delta_{j-1}\leq 0}\varphi_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2\\ &-\frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0}\varphi_{j-\frac{1}{2}}\left((1-\alpha_{j+\frac{1}{2}})+(1-\alpha_{j-\frac{1}{2}})\right)(\Delta\delta_{j-1})^2\\ &-\sum_{\Delta\delta_{j-1}\geq 0}\varphi_{j-\frac{1}{2}}\alpha_{j-\frac{3}{2}}(\Delta\delta_{j-1})^2-\frac{1}{2}\sum_{\Delta\delta_{j-1}\leq 0}\varphi_{j-\frac{1}{2}}(\alpha_{j-\frac{3}{2}}+\alpha_{j-\frac{5}{2}})(\Delta\delta_{j-1})^2,\end{aligned}\tag{C.1.44}$$

$$\mathcal{Q}_1^*:=\sum_{\Delta\delta_{j-1}\geq 0,\Delta\delta_j<0}\frac{1}{8}(\Delta\delta_{j-1})(\Delta\delta_j)-\frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0,\Delta\delta_j\leq 0}\frac{1}{8}(\Delta\delta_{j-1})^2\tag{C.1.45}$$

$$
\begin{aligned}
&- \frac{1}{2} \sum_{\Delta\delta_{j-1} \geq 0, \Delta\delta_{j-2} \leq 0} \frac{1}{8} (\Delta\delta_{j-1})^2 - \frac{1}{2} \sum_{\Delta\delta_{j-2} \geq 0, \Delta\delta_{j-1} \geq 0} \frac{1}{8} (\Delta^2\delta_{j-\frac{3}{2}})^2 \\
&+ \sum_{\Delta\delta_{j-1} \geq 0, \Delta\delta_{j} < 0} \frac{1}{8} (\Delta\delta_{j-1})(\Delta\delta_j) - \frac{1}{2} \sum_{\Delta\delta_{j-1} \leq 0, \Delta\delta_{j} \geq 0} \frac{1}{8} (\Delta\delta_{j-1})^2 \\
&- \frac{1}{2} \sum_{\Delta\delta_{j-1} \leq 0, \Delta\delta_{j-2} \geq 0} \frac{1}{8} (\Delta\delta_{j-1})^2 - \frac{1}{2} \sum_{\Delta\delta_{j-2} \leq 0, \Delta\delta_{j-1} \leq 0} \frac{1}{8} (\Delta^2\delta_{j-\frac{3}{2}})^2 \\
&- \frac{1}{32} \sum_{j=l-1}^{r+1} \left(\Delta^2 \delta_{j-\frac{3}{2}}\right)^2,
\end{aligned}
$$

and $\mathcal{E}_4, \mathcal{E}_5$, $\mathcal{E}_6$ *satisfy the following bounds*

$$
\begin{aligned}
|\mathcal{E}_4| \leq \|f_u\| \lambda^2 \gamma_2 \frac{205}{4} \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + 96 C_{u0}^2 \|f_u\| \lambda^2 \|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \\
+ \frac{1}{2} \lambda \|f_u\| \sum_{j=l-1}^{r+1} \left(\Delta^2 \delta_{j-\frac{3}{2}}\right)^2,
\end{aligned}
\tag{C.1.46}
$$

$$
\begin{aligned}
|\mathcal{E}_5| \leq \|f_u\| \lambda^2 \gamma_2 \frac{205}{4} \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + 96 C_{u0}^2 \|f_u\| \lambda^2 \|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \\
+ \frac{5}{8} \lambda \|f_u\| \sum_{j=l-1}^{r+1} \left(\Delta^2 \delta_{j-\frac{3}{2}}\right)^2,
\end{aligned}
\tag{C.1.47}
$$

and

$$
|\mathcal{E}_6| \leq \frac{1}{16} \lambda \|f_u\| \sum_{j=l-1}^{r+1} \left(\Delta^2 \delta_{j-\frac{3}{2}}\right)^2. \tag{C.1.48}
$$

Proof. Applying summation by parts, the term $\mathcal{I}_2$ in (C.1.26) can be expressed as

$$
\mathcal{I}_2 = - \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta s_{j-\frac{1}{2}} \left(\alpha_{j-\frac{3}{2}} \Delta\delta_{j-1} + (1-\alpha_{j+\frac{1}{2}}) \Delta\delta_j + \delta_{j-\frac{1}{2}} (\Delta\alpha_{j-1} - \Delta\alpha_j) \right),
$$

using which the term $\mathcal{Q}_1$ in (C.1.32) can be reformulated as

$$
\begin{aligned}
\mathcal{Q}_1 = &\sum_{j=l-1}^{r+1} \alpha_{j-\frac{3}{2}} (1-\alpha_{j-\frac{1}{2}}) (\Delta\delta_{j-\frac{1}{2}})^2 - \tilde{\mathcal{I}}_2 - \tilde{\tilde{\mathcal{I}}}_2 \\
&+ \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta s_{j-\frac{1}{2}} \delta_{j-\frac{1}{2}} (\Delta\alpha_j - \Delta\alpha_{j-1}) - \frac{1}{2} \sum_{j=l-1}^{r+1} \left(\varphi_{j-\frac{3}{2}} \Delta^2 \delta_{j-\frac{3}{2}}\right)^2,
\end{aligned}
\tag{C.1.49}
$$

where

$$
\tilde{\mathcal{I}}_2 := \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} (1-\alpha_{j+\frac{1}{2}}) \Delta\delta_j \Delta s_{j-\frac{1}{2}} \quad \text{and} \quad \tilde{\tilde{\mathcal{I}}}_2 := \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \alpha_{j-\frac{3}{2}} \Delta\delta_{j-1} \Delta s_{j-\frac{1}{2}}.
$$

Clearly, $\Delta s_{j-\frac{1}{2}} := (\Delta\delta_{j-1})_+ + (\Delta\delta_j)_-$. Utilizing this relation and following arguments similar to those in equations (50)-(53) of [144], we expand the term $\tilde{\mathcal{I}}_2$ as

$$\begin{aligned}
\tilde{\mathcal{I}}_2 = \mathcal{E}_2 &+ \sum_{\Delta\delta_{j-1}\leq 0} \varphi_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&+ \frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0} \varphi_{j-\frac{1}{2}}\left((1-\alpha_{j+\frac{1}{2}}) + (1-\alpha_{j-\frac{1}{2}})\right)(\Delta\delta_{j-1})^2 \\
&+ \sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_j<0} \varphi_{j-\frac{1}{2}}(1-\alpha_{j+\frac{1}{2}})(\Delta\delta_{j-1})(\Delta\delta_j) \\
&- \frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_j\leq 0} \varphi_{j-\frac{1}{2}}(1-\alpha_{j+\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&- \frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_{j-2}\leq 0} \varphi_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&- \frac{1}{2}\sum_{\Delta\delta_{j-2}\geq 0, \Delta\delta_{j-1}\geq 0} \varphi_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta^2\delta_{j-\frac{3}{2}})^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2 := &\sum_{\Delta\delta_{j-1}\leq 0} (\varphi_{j-\frac{3}{2}} - \varphi_{j-\frac{1}{2}})(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&+ \frac{1}{2}\sum_{\Delta\delta_{j-2}\geq 0, \Delta\delta_{j-1}\geq 0} (\varphi_{j-\frac{3}{2}} - \varphi_{j-\frac{1}{2}})(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&- \frac{1}{2}\sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_{j-2}\geq 0} (\varphi_{j-\frac{3}{2}} - \varphi_{j-\frac{1}{2}})(1-\alpha_{j-\frac{1}{2}})(\Delta^2\delta_{j-\frac{3}{2}})^2.
\end{aligned}$$

Recollecting the estimate (C.1.37), a bound can be obtained on the term $\mathcal{E}_2$ as follows:

$$\begin{aligned}
|\mathcal{E}_2| &\leq \sum_{j=l-1}^{r+1} |\varphi_{j-\frac{3}{2}} - \varphi_{j-\frac{1}{2}}| \left((\Delta\delta_{j-1})^2 + \frac{1}{2}(\Delta^2\delta_{j-\frac{3}{2}})^2\right) \qquad (C.1.50) \\
&\leq \|f_u\|\lambda^2\gamma_2\frac{5}{2}\sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}) \left((\Delta\delta_{j-1})^2 + \frac{1}{2}(\Delta^2\delta_{j-\frac{3}{2}})^2\right) \\
&\quad + 2\|f_u\|\lambda^2\|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \left((\Delta\delta_{j-1})^2 + \frac{1}{2}(\Delta^2\delta_{j-\frac{3}{2}})^2\right) \\
&\leq \|f_u\|\lambda^2\gamma_2\frac{205}{4}\sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + 96C_{u_0}^2\|f_u\|\lambda^2\|f_{uk}\|\sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}|.
\end{aligned}$$

Similarly, the term $\tilde{\tilde{\mathcal{I}}}_2$ can be expanded as

$$\begin{aligned}
\tilde{\tilde{\mathcal{I}}}_2 = \mathcal{E}_3 &+ \sum_{\Delta\delta_{j-1}\geq 0} \varphi_{j-\frac{1}{2}}\alpha_{j-\frac{3}{2}}(\Delta\delta_{j-1})^2 + \sum_{\Delta\delta_{j-1}\geq 0, \Delta\delta_j<0} \varphi_{j-\frac{1}{2}}\alpha_{j-\frac{3}{2}}(\Delta\delta_{j-1})(\Delta\delta_j) \qquad (C.1.51) \\
&+ \frac{1}{2}\sum_{\Delta\delta_{j-1}\leq 0} \varphi_{j-\frac{1}{2}}(\alpha_{j-\frac{3}{2}} + \alpha_{j-\frac{5}{2}})(\Delta\delta_{j-1})^2 - \frac{1}{2}\sum_{\Delta\delta_{j-1}\leq 0, \Delta\delta_j\geq 0} \varphi_{j-\frac{1}{2}}\alpha_{j-\frac{3}{2}}(\Delta\delta_{j-1})^2
\end{aligned}$$

$$
-\frac{1}{2}\sum_{\Delta\delta_{j-1}\le 0,\Delta\delta_{j-2}\ge 0}\varphi_{j-\frac{1}{2}}\alpha_{j-\frac{5}{2}}(\Delta\delta_{j-1})^2-\frac{1}{2}\sum_{\Delta\delta_{j-2}\le 0,\Delta\delta_{j-1}\le 0}\varphi_{j-\frac{1}{2}}\alpha_{j-\frac{5}{2}}(\Delta^2\delta_{j-\frac{3}{2}})^2,
$$

where the term $\mathcal{E}_3$ admits an estimate of the form:

$$
|\mathcal{E}_3|\le\|f_u\|\lambda^2\gamma_2\frac{205}{4}\sum_{j=l-1}^{r+1}(\delta_{j-\frac{1}{2}})^3+96C_{u0}^2\|f_u\|\lambda^2\|f_{uk}\|\sum_{j=l-1}^{r+1}|\Delta k_{j-\frac{3}{2}}|. \tag{C.1.52}
$$

It is immediate to see that

$$
\left|\alpha_{j-\frac{1}{2}}-\frac{1}{2}\right|\le\lambda\|f_u\|\quad\text{and}\quad\left|\varphi_{j-\frac{1}{2}}-\frac{1}{4}\right|\le\lambda^2\|f_u\|^2. \tag{C.1.53}
$$

In view of this estimate, we add and subtract $\frac{1}{2}$ to $\alpha_{j-\frac{1}{2}}$ and $\frac{1}{4}$ to $\varphi_{j-\frac{1}{2}}$, thereby simplifying $\tilde{\mathcal{I}}_2$ and $\tilde{\tilde{\mathcal{I}}}_2$, as given below

$$
\begin{aligned}
\tilde{\mathcal{I}}_2 =& \sum_{\Delta\delta_{j-1}\le 0}\varphi_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})(\Delta\delta_{j-1})^2 \\
&+\frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0}\varphi_{j-\frac{1}{2}}\left((1-\alpha_{j+\frac{1}{2}})+(1-\alpha_{j-\frac{1}{2}})\right)(\Delta\delta_{j-1})^2 \\
&+\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_j<0}\frac{1}{8}(\Delta\delta_{j-1})(\Delta\delta_j)-\frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_j\le 0}\frac{1}{8}(\Delta\delta_{j-1})^2 \\
&-\frac{1}{2}\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_{j-2}\le 0}\frac{1}{8}(\Delta\delta_{j-1})^2-\frac{1}{2}\sum_{\Delta\delta_{j-2}\ge 0,\Delta\delta_{j-1}\ge 0}\frac{1}{8}(\Delta^2\delta_{j-\frac{3}{2}})^2+\mathcal{E}_4,
\end{aligned} \tag{C.1.54}
$$

where the term $\mathcal{E}_4$ can be bounded as follows

$$
|\mathcal{E}_4|\le|\mathcal{E}_2|+\frac{1}{2}\lambda\|f_u\|\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2, \tag{C.1.55}
$$

on which an application of (C.1.50) yields the estimate (C.1.46). Analogous substitutions for the term $\tilde{\tilde{\mathcal{I}}}_2$ in (C.1.51) yields

$$
\begin{aligned}
\tilde{\tilde{\mathcal{I}}}_2 =& \sum_{\Delta\delta_{j-1}\ge 0}\varphi_{j-\frac{1}{2}}\alpha_{j-\frac{3}{2}}(\Delta\delta_{j-1})^2+\frac{1}{2}\sum_{\Delta\delta_{j-1}\le 0}\varphi_{j-\frac{1}{2}}(\alpha_{j-\frac{3}{2}}+\alpha_{j-\frac{5}{2}})(\Delta\delta_{j-1})^2 \\
&+\sum_{\Delta\delta_{j-1}\ge 0,\Delta\delta_j<0}\frac{1}{8}(\Delta\delta_{j-1})(\Delta\delta_j)-\frac{1}{2}\sum_{\Delta\delta_{j-1}\le 0,\Delta\delta_j\ge 0}\frac{1}{8}(\Delta\delta_{j-1})^2 \\
&-\frac{1}{2}\sum_{\Delta\delta_{j-1}\le 0,\Delta\delta_{j-2}\ge 0}\frac{1}{8}(\Delta\delta_{j-1})^2-\frac{1}{2}\sum_{\Delta\delta_{j-2}\le 0,\Delta\delta_{j-1}\le 0}\frac{1}{8}(\Delta^2\delta_{j-\frac{3}{2}})^2+\mathcal{E}_5,
\end{aligned} \tag{C.1.56}
$$

where the term $\mathcal{E}_5$ can be bounded as follows

$$
|\mathcal{E}_5|\le|\mathcal{E}_3|+\frac{5}{8}\lambda\|f_u\|\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right), \tag{C.1.57}
$$

which combined with (C.1.52) gives the estimate (C.1.47). Further, adding and subtracting $\dfrac{1}{4}$ to $\varphi_{j-\frac{3}{2}}$ in the last term of (C.1.49) and using the expressions (C.1.54) and (C.1.56), we represent the term $\mathcal{Q}_1$ as follows

$$\mathcal{Q}_1 = \mathcal{R}_1 + \mathcal{Q}_1^* - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6, \tag{C.1.58}$$

where $\mathcal{E}_6 := -\dfrac{1}{2}\displaystyle\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}} - \frac{1}{4}\right)^2\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2$. Using the CFL condition (5.5.1), we can bound $\mathcal{E}_6$ as follows:

$$|\mathcal{E}_6| \leq \frac{1}{2}(\lambda\|f_u\|)^4\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \leq \frac{1}{16}\lambda\|f_u\|\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2. \tag{C.1.59}$$

□

Lemma C.1.7. *Let $\{u_j\}_{j\in\mathbb{Z}}$ be as in Lemma C.1.1. The term $\mathcal{Q}_2$ in* (C.1.32) *can be expressed as*

$$\mathcal{Q}_2 = \mathcal{Q}_2^* + \mathcal{E}_7, \tag{C.1.60}$$

where $\mathcal{Q}_2^* := \dfrac{1}{64}\left(2\displaystyle\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 - \sum_{j=l-1}^{r+1}\left(\Delta^2 s_{j-1}\right)^2\right)$ *and*

$$|\mathcal{E}_7| \leq \frac{1}{8}\lambda\|f_u\|\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2. \tag{C.1.61}$$

Further, for $\mathcal{Q}_1^*$ as in (C.1.6)*, we have the lower bound*

$$\mathcal{Q}_1^* + \mathcal{Q}_2^* \geq \frac{1}{2048}\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2. \tag{C.1.62}$$

Proof. Adding and subtracting $\frac{1}{4}$ to $\varphi_{j-\frac{3}{2}}$, the identity (C.1.60) is immediate, where

$$\mathcal{E}_7 := \frac{1}{4}\left(2\sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}} - \frac{1}{4}\right)^2\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 - \sum_{j=l-1}^{r+1}\left(\varphi_{j-\frac{3}{2}} - \frac{1}{4}\right)^2\left(\Delta^2 s_{j-1}\right)^2\right). \tag{C.1.63}$$

Further, invoking Lemma 4 from [144], we have

$$\sum_{j=l-1}^{r+1}\left(\Delta^2 s_{j-1}\right)^2 \leq 2\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2,$$

which, when applied to (C.1.63), implies that

$$|\mathcal{E}_7| \leq \lambda^4\|f_u\|^4\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \leq \frac{1}{8}\lambda\|f_u\|\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2, \tag{C.1.64}$$

where the last inequality follows from the CFL condition (5.5.1). Furthermore, following the proof of Lemma 4 in [144], we obtain the desired lower bound (C.1.62), thus completing the proof. □

Towards our objective of getting a lower bound on $\mathcal{D}$, we now reformulate the term R_1 from (C.1.44). By applying summation by parts and subsequently adding and subtracting $\varphi_{j-\frac{1}{2}}$ to $\varphi_{j-\frac{3}{2}}$, we rewrite the second term of (C.1.44) as

$$\begin{aligned}
&\sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta s_{j-\frac{1}{2}} \left(\delta_{j-\frac{1}{2}} (\Delta\alpha_j - \Delta\alpha_{j-1}) \right) \qquad \text{(C.1.65)} \\
&= - \sum_{j=l-1}^{r+1} \Delta\alpha_{j-1} \left(\varphi_{j-\frac{1}{2}} \Delta s_{j-\frac{1}{2}} \delta_{j-\frac{1}{2}} - \varphi_{j-\frac{3}{2}} \Delta s_{j-\frac{3}{2}} \delta_{j-\frac{3}{2}} \right) \\
&= - \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta\alpha_{j-1} \left(\Delta s_{j-\frac{1}{2}} \delta_{j-\frac{1}{2}} - \Delta s_{j-\frac{3}{2}} \delta_{j-\frac{3}{2}} \right) + \mathcal{E}_8,
\end{aligned}$$

where $\mathcal{E}_8 := \sum_{j=l-1}^{r+1} \Delta\alpha_{j-1} \left(\varphi_{j-\frac{3}{2}} - \varphi_{j-\frac{1}{2}} \right) \Delta s_{j-\frac{3}{2}} \delta_{j-\frac{3}{2}}$, and is bounded as follows

$$|\mathcal{E}_8| \leq 15\lambda^3 \|f_u\|^2 \gamma_2 \sum_{j=l-1}^{r+1} \left(\delta_{j-\frac{3}{2}} \right)^3 + 16\lambda^3 \|f_u\|^2 \|f_{uk}\| C_{u_0}^2 \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}|. \qquad \text{(C.1.66)}$$

Now, in view of (C.1.65), (C.1.44) can be written as

$$\begin{aligned}
\mathcal{R}_1 = \mathcal{E}_8 &+ \sum_{j=l-1}^{r+1} \alpha_{j-\frac{3}{2}} (1 - \alpha_{j-\frac{1}{2}}) (\Delta\delta_{j-1})^2 \qquad \text{(C.1.67)} \\
&+ \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta\alpha_{j-1} \left(\Delta s_{j-\frac{3}{2}} \delta_{j-\frac{3}{2}} - \Delta s_{j-\frac{1}{2}} \delta_{j-\frac{1}{2}} \right) \\
&- \sum_{\Delta\delta_{j-1} \leq 0} \varphi_{j-\frac{1}{2}} \left(1 - \alpha_{j-\frac{1}{2}} + \frac{\alpha_{j-\frac{3}{2}} + \alpha_{j-\frac{5}{2}}}{2} \right) (\Delta\delta_{j-1})^2 \\
&- \sum_{\Delta\delta_{j-1} \geq 0} \varphi_{j-\frac{1}{2}} \left(1 + \alpha_{j-\frac{3}{2}} - \frac{\alpha_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}}}{2} \right) (\Delta\delta_{j-1})^2.
\end{aligned}$$

In (C.1.67), adding and subtracting $\alpha_{j-\frac{3}{2}}$ to $\alpha_{j-\frac{5}{2}}$ in the fourth term and $\alpha_{j-\frac{1}{2}}$ to $\alpha_{j+\frac{1}{2}}$ in the fifth term, we write

$$\mathcal{R}_1 = \mathcal{E}_8 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5, \qquad \text{(C.1.68)}$$

where

$$\begin{aligned}
\mathcal{R}_2 &:= \sum_{j=l-1}^{r+1} \varphi_{j-\frac{1}{2}} \Delta\alpha_{j-1} \left(\Delta s_{j-\frac{3}{2}} \delta_{j-\frac{3}{2}} - \Delta s_{j-\frac{1}{2}} \delta_{j-\frac{1}{2}} \right), \\
\mathcal{R}_3 &:= \sum_{j=l-1}^{r+1} \left(\alpha_{j-\frac{3}{2}} (1 - \alpha_{j-\frac{1}{2}}) - \varphi_{j-\frac{1}{2}} (1 - \Delta\alpha_{j-1}) \right) (\Delta\delta_{j-1})^2, \\
\mathcal{R}_4 &:= \sum_{\Delta\delta_{j-1} \leq 0} \varphi_{j-\frac{1}{2}} \left(\frac{\alpha_{j-\frac{3}{2}} - \alpha_{j-\frac{5}{2}}}{2} \right) (\Delta\delta_{j-1})^2 \quad \text{and} \qquad \text{(C.1.69)} \\
\mathcal{R}_5 &:= \sum_{\Delta\delta_{j-1} \geq 0} \varphi_{j-\frac{1}{2}} \left(\frac{\alpha_{j+\frac{1}{2}} - \alpha_{j-\frac{1}{2}}}{2} \right) (\Delta\delta_{j-1})^2.
\end{aligned}$$

Next, $\mathcal{R}_4$ admits a lower bound

$$\mathcal{R}_4 \geq -16(C_{u_0})^2 \lambda \|f_{ku}\| \sum_{j=l-2}^{r} |\Delta k_{j-\frac{3}{2}}| =: \tilde{\mathcal{R}}_4. \tag{C.1.70}$$

This holds true because, by (C.1.42), we have

$$\begin{aligned}
\mathcal{R}_4 &\geq \sum_{\Delta\delta_{j-1}\leq 0} \frac{\varphi_{j-\frac{1}{2}}}{2} \left(\frac{3}{8}\lambda\gamma_1 \left(\delta^n_{j-\frac{3}{2}} + \delta^n_{j-\frac{5}{2}} \right) + \lambda \Delta k_{j-\frac{5}{2}} f_{ku}(\bar{k}_{j-\frac{5}{2}}, \bar{u}_{j-\frac{3}{2}}) \right) (\Delta\delta_{j-1})^2 \\
&\geq \frac{3}{16}\lambda\gamma_1 \sum_{\Delta\delta_{j-1}\leq 0} \varphi_{j-\frac{1}{2}} \left(\delta^n_{j-\frac{3}{2}} + \delta^n_{j-\frac{5}{2}} \right) (\Delta\delta_{j-1})^2 \\
&\quad + \sum_{\Delta\delta_{j-1}\leq 0} \lambda \Delta k_{j-\frac{5}{2}} f_{ku}(\bar{k}_{j-\frac{5}{2}}, \bar{u}_{j-\frac{3}{2}}) (\Delta\delta_{j-1})^2 \\
&\geq \sum_{\Delta\delta_{j-1}\leq 0} \lambda \Delta k_{j-\frac{5}{2}} f_{ku}(\bar{k}_{j-\frac{5}{2}}, \bar{u}_{j-\frac{3}{2}}) (\Delta\delta_{j-1})^2,
\end{aligned}$$

and also by noting the fact that $(\Delta\delta_{j-1})^2 \leq 16(C_{u_0})^2$. Similar arguments on $\mathcal{R}_5$ yield

$$\mathcal{R}_5 \geq -16(C_{u_0})^2 \lambda \|f_{ku}\| \sum_{j=l}^{r+2} |\Delta k_{j-\frac{3}{2}}| =: \tilde{\mathcal{R}}_5 \tag{C.1.71}$$

Now, using Lemmas C.1.6 and C.1.7, along with with the expression (C.1.68) in (C.1.31), we write

$$\mathcal{D} = \mathcal{Q}_1^* + \mathcal{Q}_2^* + \mathcal{Q}_3^* + \mathcal{R}_4 + \mathcal{R}_5 + P_2 + \mathcal{E}_9, \tag{C.1.72}$$

where $Q_3^* := \mathcal{Q}_3 + \mathcal{R}_2 + \mathcal{R}_3$ and $\mathcal{E}_9 := \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 + \mathcal{E}_7 + \mathcal{E}_8$. Further, combining the expressions (C.1.38),(C.1.46), (C.1.47), (C.1.59), (C.1.61) and (C.1.66), using the CFL condition (5.5.1) and after suitable algebraic manipulations, we obtain the following bound

$$\begin{aligned}
\mathcal{E}_9 \geq{}& -255\lambda^2 \|f_u\| \gamma_2 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 - \frac{21}{16}\lambda\|f_u\| \sum_{j=l-1}^{r+1} \left(\Delta^2 \delta_{j-\frac{3}{2}}\right)^2 \\
&- \left(24 + 192\|f_u\| + 16\lambda\|f_u\|^2\right) C_{u_0}^2 \lambda^2 \|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}|.
\end{aligned} \tag{C.1.73}$$

Next, we establish a lower bound on the term $\mathcal{Q}_3^*$.

Lemma C.1.8. *The term* $\mathcal{Q}_3^*$ *in* (C.1.72) *admits the lower bound*

$$\mathcal{Q}_3^* \geq \frac{3}{64} \sum_{j=l-1}^{r+1} \lambda\gamma_1 (\delta_{j-\frac{1}{2}})^3 - 6\lambda\|f_{ku}\| C_{u_0}^2 \sum_{j=l-2}^{r+1} |\Delta k_{j-\frac{1}{2}}| + \mathcal{E}_{10}, \tag{C.1.74}$$

where C_{u_0} *is as in* (5.4.3) *and* $\mathcal{E}_{10}$ *is bounded as follows*

$$|\mathcal{E}_{10}| \leq (28\lambda\|f_u\|) \lambda \|f_{uk}\| C_{u_0}^2 \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| + \left(\frac{35}{2}\lambda\|f_u\|\right) \lambda\gamma_2 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3.$$

Proof. Recalling the notation $\varphi_{j-\frac{1}{2}} := \alpha_{j-\frac{1}{2}}(1-\alpha_{j-\frac{1}{2}})$ and using it in $\mathcal{R}_3$ of (C.1.69), we write $\mathcal{Q}_3^*$ as

$$\mathcal{Q}_3^* = \mathcal{Q}_3 + \mathcal{R}_2 - \sum_{j=l-1}^{r+1} (1-\alpha_{j-\frac{1}{2}})^2(\alpha_{j-\frac{1}{2}} - \alpha_{j-\frac{3}{2}})(\Delta\delta_{j-1})^2.$$

Writing $\varphi_{j-\frac{1}{2}} = \frac{1}{4} + (\varphi_{j-\frac{1}{2}} - \frac{1}{4})$, $\alpha_{j-\frac{3}{2}} = \frac{1}{2} + (\alpha_{j-\frac{3}{2}} - \frac{1}{2})$ and $\alpha_{j-\frac{1}{2}} = \frac{1}{2} + (\alpha_{j-\frac{1}{2}} - \frac{1}{2})$, the term $\mathcal{Q}_3^*$ can be expressed as

$$\begin{aligned}
\mathcal{Q}_3^* :=& \sum_{j=l-1}^{r+1} \Delta\alpha_{j-1}\left(\frac{1}{2}(\delta_{j-\frac{3}{2}})^2 + \frac{1}{2}(\delta_{j-\frac{1}{2}})^2\right) \\
&+ \sum_{j=l-1}^{r+1} \frac{1}{4}\Delta\alpha_{j-1}\left(\Delta s_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}} - \Delta s_{j-\frac{1}{2}}\delta_{j-\frac{1}{2}}\right) - \sum_{j=l-1}^{r+1}\frac{1}{4}\Delta\alpha_{j-1}(\Delta\delta_{j-1})^2 + \mathcal{E}_{10} \\
=& \frac{1}{4}\sum_{j=l-1}^{r+1} \Delta\alpha_{j-1} z_{j-1} + \mathcal{E}_{10},
\end{aligned} \tag{C.1.75}$$

where

$$\begin{aligned}
z_{j-1} &:= 2(\delta_{j-\frac{3}{2}})^2 + 2(\delta_{j-\frac{1}{2}})^2 + \Delta s_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}} - \Delta s_{j-\frac{1}{2}}\delta_{j-\frac{1}{2}} - (\Delta\delta_{j-1})^2 \quad \text{and} \\
\mathcal{E}_{10} &:= \sum_{j=l-1}^{r+1} \Delta\alpha_{j-1}\Bigg[\left(\alpha_{j-\frac{3}{2}} - \frac{1}{2}\right)(\delta^n_{j-\frac{3}{2}})^2 + \left(\frac{1}{2} - \alpha_{j-\frac{1}{2}}\right)(\delta^n_{j-\frac{1}{2}})^2 \\
&\quad + \left(\varphi_{j-\frac{1}{2}} - \frac{1}{4}\right)\left(\Delta s_{j-\frac{3}{2}}\delta_{j-\frac{3}{2}} - \Delta s_{j-\frac{1}{2}}\delta_{j-\frac{1}{2}}\right) \\
&\quad - \left(\frac{3}{4} + (\alpha_{j-\frac{1}{2}})^2 - 2\alpha_{j-\frac{1}{2}}\right)(\Delta\delta_{j-1})^2\Bigg].
\end{aligned}$$

Recalling the notation $\alpha_{j-\frac{1}{2}} := \frac{1}{2} + \lambda\bar{a}_{j-\frac{1}{2}}$, it is clear from (C.1.35) and (C.1.36) that $\Delta\alpha_{j-1}$ admits a bound $|\Delta\alpha_{j-1}| \leq \lambda\|f_{uk}\||\Delta k_{j-\frac{3}{2}}| + \frac{5}{4}\lambda\gamma_2(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}})$. Consequently, we obtain the following bound on $\mathcal{E}_{10}$:

$$\begin{aligned}
|\mathcal{E}_{10}| \leq& \left(\frac{7}{2}\lambda\|f_u\|\right)\sum_{j=l-1}^{r+1}\lambda\|f_{uk}\||\Delta k_{j-\frac{3}{2}}|\left((\delta_{j-\frac{1}{2}})^2 + (\delta_{j-\frac{3}{2}})^2\right) \\
&+ \left(\frac{7}{2}\lambda\|f_u\|\right)\sum_{j=l-1}^{r+1}\frac{5}{4}\lambda\gamma_2(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}})\left((\delta_{j-\frac{1}{2}})^2 + (\delta_{j-\frac{3}{2}})^2\right) \\
\leq& \left(\frac{7}{2}\lambda\|f_u\|\right)8\lambda\|f_{uk}\|C_{u_0}^2\sum_{j=l-1}^{r+1}|\Delta k_{j-\frac{3}{2}}| + \left(\frac{7}{2}\lambda\|f_u\|\right)\frac{5}{4}\lambda\gamma_2 6\sum_{j=l-1}^{r+1}(\delta_{j-\frac{1}{2}})^3 \\
\leq& \left(28\lambda\|f_u\|\right)\lambda\|f_{uk}\|C_{u_0}^2\sum_{j=l-1}^{r+1}|\Delta k_{j-\frac{3}{2}}| + \left(\frac{105}{4}\lambda\|f_u\|\right)\lambda\gamma_2\sum_{j=l-1}^{r+1}(\delta_{j-\frac{1}{2}})^3.
\end{aligned} \tag{C.1.76}$$

Next, to obtain a lower bound on $\mathcal{Q}_3^*$, we need a bound on the first term in the RHS of

(C.1.75). To this end, we first make the following observation on z_{j-1} in (C.1.75):

$$\begin{aligned} z_{j-1} &= \left(\delta_{j-\frac{3}{2}} + \delta_{j-\frac{1}{2}}\right)^2 + s_{j-1}(\delta_{j-\frac{3}{2}} + \delta_{j-\frac{1}{2}}) - \delta_{j-\frac{3}{2}} s_{j-2} - \delta_{j-\frac{1}{2}} s_j \\ &\geq 2\delta_{j-\frac{1}{2}}\delta_{j-\frac{3}{2}} + s_{j-1}(\delta_{j-\frac{3}{2}} + \delta_{j-\frac{1}{2}}). \end{aligned} \tag{C.1.77}$$

This establishes that $z_{j-1} \geq 0$. Furthermore, combining the expression (C.1.77) with the L^∞-stability (5.4.3) of the scheme (5.2.4), we deduce that $|z_{j-1}| \leq 24C_{u_0}^2$, for $j \in \mathbb{Z}$. Next, we obtain a bound

$$\Delta\alpha_{j-1} z_{j-1} + \Delta\alpha_{j-2} z_{j-2} \geq \frac{3}{8}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C_{u_0}^2\lambda\|f_{ku}\|(|\Delta k_{j-\frac{3}{2}}| + |\Delta k_{j-\frac{5}{2}}|), \tag{C.1.78}$$

which is verified by considering the two possible cases.

Case 1 ($\delta_{j-\frac{1}{2}} \leq \delta_{j-\frac{3}{2}}$ and $\delta_{j-\frac{5}{2}} \leq \delta_{j-\frac{3}{2}}$): In this case, as $s_{j-2} = \delta_{j-\frac{5}{2}}$, from the definition of z_{j-1}, it follows that

$$\begin{aligned} z_{j-1} &= (\delta_{j-\frac{3}{2}})^2 + (\delta_{j-\frac{1}{2}})^2 + 2\delta_{j-\frac{1}{2}}\delta_{j-\frac{3}{2}} + s_{j-1}(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}) - \delta_{j-\frac{3}{2}} s_{j-2} - \delta_{j-\frac{1}{2}} s_j \\ &\geq (\delta_{j-\frac{3}{2}})^2 - \delta_{j-\frac{3}{2}}\delta_{j-\frac{5}{2}}. \end{aligned}$$

This bound, combined with (C.1.42) implies that

$$\begin{aligned} \Delta\alpha_{j-1} z_{j-1} &\geq \frac{3}{8}\lambda\gamma_1\left(\delta_{j-\frac{1}{2}} + \delta_{j-\frac{3}{2}}\right)\left((\delta_{j-\frac{3}{2}})^2 - \delta_{j-\frac{3}{2}}\delta_{j-\frac{5}{2}}\right) - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{3}{2}}| \\ &\geq \frac{3}{8}\lambda\gamma_1\left((\delta_{j-\frac{3}{2}})^3 - (\delta_{j-\frac{3}{2}})^2\delta_{j-\frac{5}{2}}\right) - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{3}{2}}|. \end{aligned}$$

Furthermore, in this case, we also have $z_{j-2} \geq 2\delta_{j-\frac{5}{2}}\delta_{j-\frac{3}{2}}$, which yields

$$\begin{aligned} \Delta\alpha_{j-2} z_{j-2} &\geq \frac{3}{8}\lambda\gamma_1 2\delta_{j-\frac{5}{2}}\delta_{j-\frac{3}{2}}\left(\delta^n_{j-\frac{3}{2}} + \delta^n_{j-\frac{5}{2}}\right) - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{5}{2}}| \\ &\geq \frac{3}{4}\lambda\gamma_1\delta_{j-\frac{5}{2}}(\delta_{j-\frac{3}{2}})^2 - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{5}{2}}|. \end{aligned}$$

This leads to the conclusion

$$\Delta\alpha_{j-1} z_{j-1} + \Delta\alpha_{j-2} z_{j-2} \geq \frac{3}{8}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C_{u_0}^2\lambda\|f_{ku}\|(|\Delta k_{j-\frac{3}{2}}| + |\Delta k_{j-\frac{5}{2}}|), \tag{C.1.79}$$

for Case 1.

Case 2 ($\delta_{j-\frac{1}{2}} \geq \delta_{j-\frac{3}{2}}$ or $\delta_{j-\frac{5}{2}} \geq \delta_{j-\frac{3}{2}}$): In the case when $\delta_{j-\frac{1}{2}} \geq \delta_{j-\frac{3}{2}}$, we have $s_{j-1} = \delta_{j-\frac{3}{2}}$. Consequently, applying (C.1.77) yields $z_{j-1} \geq 4(\delta_{j-\frac{3}{2}})^2$. Further, using (C.1.42), we can write

$$\begin{aligned} \Delta\alpha_{j-1} z_{j-1} &\geq \frac{3}{8}\lambda\gamma_1 4(\delta_{j-\frac{3}{2}})^2\left(\delta^n_{j-\frac{1}{2}} + \delta^n_{j-\frac{3}{2}}\right) - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{3}{2}}| \\ &\geq \frac{3}{2}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{3}{2}}|. \end{aligned}$$

Moreover, (C.1.42) also implies that $\Delta\alpha_{j-2} z_{j-2} \geq -24C_{u_0}^2\lambda\|f_{ku}\||\Delta k_{j-\frac{5}{2}}|$. Thus,

$$\Delta\alpha_{j-1} z_{j-1} + \Delta\alpha_{j-2} z_{j-2} \geq \frac{3}{4}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C_{u_0}^2\lambda\|f_{ku}\|(|\Delta k_{j-\frac{3}{2}}| + |\Delta k_{j-\frac{5}{2}}|), \tag{C.1.80}$$

for the case when $\delta_{j-\frac{1}{2}} \geq \delta_{j-\frac{3}{2}}$. Next, for the case when $\delta_{j-\frac{5}{2}} \geq \delta_{j-\frac{3}{2}}$, we have $s_{j-2} = \delta_{j-\frac{3}{2}}$. Therefore, using (C.1.77), we have $z_{j-2} \geq 4(\delta_{j-\frac{3}{2}})^2$. Applying (C.1.42) again, we obtain

$$\begin{aligned}\Delta\alpha_{j-2} z_{j-2} &\geq \frac{3}{8}\lambda\gamma_1 4(\delta_{j-\frac{3}{2}})^2 \left(\delta^n_{j-\frac{3}{2}} + \delta^n_{j-\frac{5}{2}}\right) - 24C^2_{u0}\lambda\|f_{ku}\||\Delta k_{j-\frac{5}{2}}| \\ &\geq \frac{3}{2}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C^2_{u0}\lambda\|f_{ku}\||\Delta k_{j-\frac{5}{2}}|.\end{aligned}$$

Again, using (C.1.77) it follows that $\Delta\alpha_{j-1} z_{j-1} \geq -24C^2_{u0}\lambda\|f_{ku}\||\Delta k_{j-\frac{3}{2}}|$, leading to

$$\Delta\alpha_{j-1} z_{j-1} + \Delta\alpha_{j-2} z_{j-2} \geq \frac{3}{4}\lambda\gamma_1(\delta_{j-\frac{3}{2}})^3 - 24C^2_{u0}\lambda\|f_{ku}\|(|\Delta k_{j-\frac{3}{2}}| + |\Delta k_{j-\frac{5}{2}}|), \tag{C.1.81}$$

thus concluding Case 2.

Combining both the cases, the estimates (C.1.79),(C.1.80) and (C.1.81) establish the bound (C.1.78). Furthermore, using (C.1.78), we obtain

$$\begin{aligned}\sum_{j=l-1}^{r+1} \Delta\alpha_{j-1} z_{j-1} &= \frac{1}{2}\left(\sum_{j=l}^{r+1} \Delta\alpha_{j-1} z_{j-1} + \sum_{j=l+1}^{r+2} \Delta\alpha_{j-2} z_{j-2}\right) \\ &= \frac{1}{2}\left(\sum_{j=l}^{r+2} (\Delta\alpha_{j-1} z_{j-1} + \Delta\alpha_{j-2} z_{j-2})\right) \\ &\geq \frac{3}{16}\sum_{j=l-1}^{r+1} \lambda\gamma_1(\delta_{j-\frac{1}{2}})^3 - 24\lambda\|f_{ku}\|C^2_{u0}\sum_{j=l-2}^{r+1}|\Delta k_{j-\frac{1}{2}}|.\end{aligned} \tag{C.1.82}$$

The bound (C.1.82) together with (C.1.75) yields the desired estimate (C.1.74) on Q^*_3. $\square$

Finally, to conclude this section on non-decreasing sequences, we provide the proof of Lemma C.1.1.

Proof of Lemma C.1.1. By invoking the bound (C.1.62) for $\mathcal{Q}^*_1 + \mathcal{Q}^*_2$ from Lemma C.1.7 and the bound (C.1.74) for $\mathcal{Q}^*_3$ from Lemma C.1.8, in the expression (C.1.72), we obtain

$$\begin{aligned}\mathcal{D} \geq &\frac{3}{64}\sum_{j=l-1}^{r+1} \lambda\gamma_1(\delta_{j-\frac{1}{2}})^3 + \frac{1}{2048}\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 + \mathcal{E}_9 + \mathcal{E}_{10} \\ &+\tilde{\mathcal{R}}_4 + \tilde{\mathcal{R}}_5 + P_2 - 6\lambda\|f_{ku}\|C^2_{u0}\sum_{j=l-2}^{r+1}|\Delta k_{j-\frac{1}{2}}|.\end{aligned} \tag{C.1.83}$$

Now, using the bounds (C.1.73) and (C.1.76) for $\mathcal{E}_9$ and $\mathcal{E}_{10}$, respectively, and the CFL condition (5.5.1), we further simplify (C.1.83) as follows

$$\mathcal{D} \geq \frac{3}{64}\lambda\gamma_1\sum_{j=l-1}^{r+1}(\delta_{j-\frac{1}{2}})^3 + \frac{1}{2048}\sum_{j=l-1}^{r+1}\left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \tag{C.1.84}$$

$$
\begin{aligned}
&- \frac{1125}{4}\lambda^2\|f_u\|\gamma_2 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 - \frac{21}{16}\lambda\|f_u\| \sum_{j=l-1}^{r+1} \left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \\
&- \left(24 + 220\|f_u\| + 16\lambda\|f_u\|^2\right) C_{u0}^2 \lambda^2\|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \\
&+ \tilde{\mathcal{R}}_4 + \tilde{\mathcal{R}}_5 + P_2 - 6\lambda\|f_{ku}\|C_{u0}^2 \sum_{j=l-2}^{r+1} |\Delta k_{j-\frac{1}{2}}| \\
\geq\ & \frac{15}{1600}\lambda\gamma_1 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + \frac{1}{6400} \sum_{j=l-1}^{r+1} \left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \\
&- \left(24 + 220\|f_u\| + 16\lambda\|f_u\|^2\right) C_{u0}^2 \lambda^2\|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \\
&+ \tilde{\mathcal{R}}_4 + \tilde{\mathcal{R}}_5 + P_2 - 6\lambda\|f_{ku}\|C_{u0}^2 \sum_{j=l-2}^{r+1} |\Delta k_{j-\frac{1}{2}}|,
\end{aligned}
$$

where the last inequality follows from the CFL condition (5.5.1). Adding and subtracting $\sum_{j=l-1}^{r+1}(\delta'_{j-1})^2$ to $\mathcal{D}$ in (C.1.84) and subsequently using (C.1.10) and the CFL condition (5.5.1), we have

$$
\begin{aligned}
\sum_{j=l-1}^{r+1} \left((\delta_{j-\frac{1}{2}})^2 - (\delta'_{j-1})^2\right) &\geq \sum_{j=l-1}^{r+1} \left((\delta_{j-\frac{1}{2}})^2 - (\delta''_{j-1})^2\right) - \left|\sum_{j=l-1}^{r+1} (\delta'_{j-1})^2 - (\delta''_{j-1})^2\right| \\
&\geq \frac{1}{500}\lambda\gamma_1 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + \frac{1}{6400} \sum_{j=l-1}^{r+1} \left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 - P_1 \\
&\quad - \left(24 + 220\|f_u\| + 16\lambda\|f_u\|^2\right) C_{u_0}^2 \lambda^2\|f_{uk}\| \sum_{j=l-1}^{r+1} |\Delta k_{j-\frac{3}{2}}| \\
&\quad + \tilde{\mathcal{R}}_4 + \tilde{\mathcal{R}}_5 + P_2 - 6\lambda\|f_{ku}\|C_{u_0}^2 \sum_{j=l-2}^{r+1} |\Delta k_{j-\frac{1}{2}}|.
\end{aligned}
\tag{C.1.85}
$$

Further, substituting the values of $P_1, \tilde{\mathcal{R}}_4$ and $\tilde{\mathcal{R}}_5$ from (C.1.10), (C.1.70) and (C.1.71), respectively, and using the estimate (C.1.28) for P_2, we write

$$
\begin{aligned}
\sum_{j=l-1}^{r+1} \left((\delta_{j-\frac{1}{2}})^2 - (\delta'_{j-1})^2\right) &\geq \frac{1}{500}\lambda\gamma_1 \sum_{j=l-1}^{r+1} (\delta_{j-\frac{1}{2}})^3 + \frac{1}{6400} \sum_{j=l-1}^{r+1} \left(\Delta^2\delta_{j-\frac{3}{2}}\right)^2 \\
&\quad - \Theta \sum_{j=l-2}^{r+2} |\Delta k_{j-\frac{3}{2}}|,
\end{aligned}
\tag{C.1.86}
$$

where

$$
\begin{aligned}
\Theta := {}& 24\lambda^2(C_{u_0})^2\|f_{uk}\| + 38\lambda(C_{u_0})^2\|f_{ku}\| + \left(236(C_{u_0})^2 + 16\lambda\|f_u\|\right)\lambda^2\|f_u\|\|f_{uk}\| \\
&+ \left(16\lambda^2 C_{u_0}\|k\| + 44\lambda^2(C_{u_0})^2\gamma_2\|f_u\| + 8\lambda\|f_k\|\|k\| + 24C_{u_0}\right)\lambda\|f_k\|.
\end{aligned}
$$

Finally, we obtain the desired estimate (C.1.1) since

$$\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta'_{j-1})_+^2\right)\geq\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta'_{j-1})^2\right).$$

$\square$

A summary of the proof of Lemma C.1.1: To derive the desired estimate for the term $\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta'_{j-1})^2\right)$, we begin by introducing the modified jumps δ''_j (see (C.1.11)) and show in Lemma C.1.3 that the term $\left|\sum_{j=l-1}^{r+1}(\delta'_{j-1})^2-\sum_{j=l-1}^{r+1}(\delta''_{j-1})^2\right|$ is appropriately bounded. With this estimate in hand, we focus on the term $\mathcal{D}:=\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta''_{j-1})^2\right)$, which we decompose as $\mathcal{D}=\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3+P_2+\mathcal{E}_1$. Next, in Lemmas C.1.6 and C.1.7, we reformulate the terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as $\mathcal{Q}_1=\mathcal{R}_1+\mathcal{Q}_1^*-\mathcal{E}_4-\mathcal{E}_5-\mathcal{E}_6$ and $\mathcal{Q}_2=\mathcal{Q}_2^*+\mathcal{E}_7$. We then rewrite the term $\mathcal{R}_1$ in $\mathcal{Q}_1$ as $\mathcal{R}_1=\mathcal{E}_8+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4+\mathcal{R}_5$ (see (C.1.68)). The above reformulations allow us to write $\mathcal{D}=\mathcal{Q}_1^*+\mathcal{Q}_2^*+\mathcal{Q}_3^*+\mathcal{R}_4+\mathcal{R}_5+P_2+\mathcal{E}_9$ (see (C.1.72)), where $\mathcal{E}_9:=\mathcal{E}_1-\mathcal{E}_4-\mathcal{E}_5-\mathcal{E}_6+\mathcal{E}_7+\mathcal{E}_8$ and $\mathcal{Q}_3^*:=\mathcal{Q}_3+\mathcal{R}_2+\mathcal{R}_3$. Now, we obtain suitable bounds for P_2 (see (C.1.28)) and $\mathcal{E}_1$ (see C.1.38). Further, through Lemmas C.1.6 and C.1.7, we show that the terms $\mathcal{E}_4,\mathcal{E}_5,\mathcal{E}_6$ and $\mathcal{E}_7$ are bounded. Lemma C.1.7 also provides a lower bound for $\mathcal{Q}_1^*+\mathcal{Q}_2^*$. In (C.1.70) and (C.1.71), we derive lower bounds for $\mathcal{R}_4$ and $\mathcal{R}_5$, respectively. Furthermore, in (C.1.73) we estimate the term $\mathcal{E}_9$, while in Lemma C.1.8, we establish a lower bound for $\mathcal{Q}_3^*$. Combining these results, we derive a lower bound for $\mathcal{D}$. Using this lower bound alongside Lemma C.1.3, we finally obtain a lower bound for the term $\sum_{j=l-1}^{r+1}\left((\delta_{j-\frac{1}{2}})^2-(\delta'_{j-1})^2\right)$, thereby completing the proof.

C.1.2 Concluding the proof of Lemma 5.5.1

First, we present an auxiliary lemma for non-increasing sequences, which will be used in the proof of Lemma 5.5.1.

Lemma C.1.9. *Consider a non-increasing sequence $\{u_j\}_{j\in\mathbb{Z}}$, with jumps $\delta_{j+\frac{1}{2}}:=u_{j+1}-u_j$. Let $\{u'_{j+\frac{1}{2}}\}_{j\in\mathbb{Z}}$ be obtained from $\{u_j\}_{j\in\mathbb{Z}}$ by applying the time-update formula* (5.2.4) *and denote the corresponding jumps $\delta'_j:=u'_{j+\frac{1}{2}}-u'_{j-\frac{1}{2}}$ for $j\in\mathbb{Z}$. Under the CFL condition* (5.5.1)*, the jump sequence $\{\delta'_j\}_{j\in\mathbb{Z}}$ satisfies the following estimate*

$$\left(\delta'_j\right)_+\leq\lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}|+|\Delta k_{j-\frac{1}{2}}|\right). \tag{C.1.87}$$

Proof. As $\delta_{j+\frac{1}{2}}\leq 0$ for $j\in\mathbb{Z}$, applying the CFL condition (5.5.1) to (C.1.3), it follows that

$$\begin{aligned}\delta'_j&\leq\left(\frac{1}{2}-\frac{1}{8}-\kappa-\kappa^2\right)(\delta_{j+\frac{1}{2}}+\delta_{j-\frac{1}{2}})+\lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}|+|\Delta k_{j-\frac{1}{2}}|\right)\\&\leq\lambda\|f_k\|\left(|\Delta k_{j+\frac{1}{2}}|+|\Delta k_{j-\frac{1}{2}}|\right),\end{aligned} \tag{C.1.88}$$

from which the estimate (C.1.87) follows directly. $\square$

Proof of Lemma 5.5.1. Denote the jumps $\delta_j^{n+1} := u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1}$ and $\delta_{j-\frac{1}{2}}^n := u_j^n - u_{j-1}^n$ for $j \in \mathbb{Z}$. The key step in deriving the estimate (5.5.2) involves decomposing $\{u_j^n\}_{j\in\mathbb{Z}}$ into monotone sequences and invoking Lemmas C.1.1 and C.1.9 to obtain useful estimates. To this end, we split the index set $\mathbb{Z}$ of the sequence $\{u_j^n\}_{j\in\mathbb{Z}}$ into maximal subsets of the form $\Gamma_m := \{l-1, l, l+1, \cdots, r\}, m \in S \subseteq \mathbb{N}$ such that $\{u_j^n\}_{j\in\Gamma_m}$ is either non-decreasing or non-increasing. Clearly, $\bigcup_{m\in S}\Gamma_m = \mathbb{Z}$. Now, for each $m \in S$, we define a new sequence $\{u_j^m\}_{j\in\mathbb{Z}}$, which is either non-decreasing or non-increasing, as follows:

$$u_j^m := \begin{cases} u_j^n, & \text{if}\, l-1 \le j \le r, \\ u_{l-1}^n, & \text{if } j < l-1, \\ u_r^n, & \text{if } j > r. \end{cases} \tag{C.1.89}$$

With this definition, we observe that for each $m \in S$, the jump sequence $\{\delta_{j+\frac{1}{2}}^m\}_{j\in\mathbb{Z}}$ associated with u_j^m is either non-negative or non-positive. Additionally,

$$\sum_{j\in\mathbb{Z}}(\delta_{j+\frac{1}{2}}^n)_+^2 = \sum_{m\in S}\sum_{j\in\mathbb{Z}}(\delta_{j+\frac{1}{2}}^m)_+^2 = \sum_{m\in S}\sum_{j\in\Gamma_m}(\delta_{j+\frac{1}{2}}^m)_+^2. \tag{C.1.90}$$

Now, our aim is to compare the sums $\sum_{j\in\mathbb{Z}}(\delta_j^{n+1})_+^2$ and $\sum_{m\in S}\sum_{j\in\mathbb{Z}}(\delta_j^{m'})_+^2$, where for $m \in S$, $\delta_j^{m'} := u_{j+\frac{1}{2}}^{m'} - u_{j-\frac{1}{2}}^{m'}$, and $\{u_{j+\frac{1}{2}}^{m'}\}_{j\in\mathbb{Z}}$ is obtained by applying the scheme (5.2.4) to $\{u_j^m\}_{j\in\mathbb{Z}}$. For this, first we note that the jumps $\{\delta_j^{n+1}\}_{j\in\mathbb{Z}}$ can be expressed as

$$\begin{aligned} \delta_j^{n+1} &= \frac{1}{2}(\delta_{j+\frac{1}{2}}^n + \delta_{j-\frac{1}{2}}^n) - \lambda\left(f(k_{j+1}, u_{j+1}^{n+\frac{1}{2}}) - 2f(k_j, u_j^{n+\frac{1}{2}}) + f(k_{j-1}, u_{j-1}^{n+\frac{1}{2}})\right) \\ &\quad - \frac{1}{8}(\sigma_{j+1}^n - 2\sigma_j^n + \sigma_{j-1}^n). \end{aligned} \tag{C.1.91}$$

Next, consider a region where the sequence $\{u_j^n\}_{j\in\mathbb{Z}}$ is monotone, say for $j \in \{l-1, l, \ldots, r\}$. In this case it is evident that $\delta_{j+\frac{1}{2}}^n = \delta_{j+\frac{1}{2}}^m$ for $j \in \{l-1, l, \ldots, r-1\}$, for some $m \in S$. Consequently, from (C.1.91) we have $\delta_j^{n+1} = \delta_j^{m'}$, for $j \in \{l, l+1, \ldots, r-1\}$. In other words, away from the local extremum points of $\{u_j^n\}_{j\in\mathbb{Z}}$, the relation $\delta_j^{n+1} = \delta_j^{m'}$ holds for some m. On the other hand, at the local extremum points, say $j \in \{l-1, r\}$, of $\{u_j^n\}_{j\in\mathbb{Z}}$ with $\{u_j^n\}$ non-decreasing on $j \in \{l-1, l, \ldots, r\}$, we have

$$\begin{aligned} \delta_r^{n+1} &= \delta_r^{m'} + \delta_r^{(m+1)'} + \lambda f_k(\bar{k}_{r+\frac{1}{2}}, u_r^n)\Delta k_{r+\frac{1}{2}} - \lambda f_k(\bar{k}_{r-\frac{1}{2}}, u_r^n)\Delta k_{r-\frac{1}{2}}, \\ \delta_{l-1}^{n+1} &= \delta_{l-1}^{(m-1)'} + \delta_{l-1}^{m'} + \lambda f_k(\bar{k}_{l-\frac{1}{2}}, u_{l-1}^n)\Delta k_{l-\frac{1}{2}} - \lambda f_k(\bar{k}_{l-\frac{3}{2}}, u_{l-1}^n)\Delta k_{l-\frac{3}{2}}, \end{aligned} \tag{C.1.92}$$

for $\bar{k}_{r-\frac{1}{2}} \in \mathcal{I}(k_{r-1}, k_r)$, $\bar{k}_{r+\frac{1}{2}} \in \mathcal{I}(k_r, k_{r+1})$, $\bar{k}_{l-\frac{1}{2}} \in \mathcal{I}(k_{l-1}, k_l)$ and $\bar{k}_{l-\frac{3}{2}} \in \mathcal{I}(k_{l-2}, k_{l-1})$.

Further, using Lemma C.1.9 for the term $(\delta_r^{(m+1)'})_+$(which is generated from the non-increasing sequence $\{u^{m+1}\}_{j\in\mathbb{Z}}$) in (C.1.92), along with the property that $a, b \in \mathbb{R}$, $(a+b)_+ \leq a_+ + b_+$, we obtain

$$\begin{aligned}\left(\delta_r^{n+1}\right)_+ &\leq (\delta_r^{m'})_+ + (\delta_r^{(m+1)'})_+ + \lambda\|f_k\|(|\Delta k_{r+\frac{1}{2}}| + |\Delta k_{r-\frac{1}{2}}|) \\ &\leq (\delta_r^{m'})_+ + 2\lambda\|f_k\|(|\Delta k_{r+\frac{1}{2}}| + |\Delta k_{r-\frac{1}{2}}|).\end{aligned} \tag{C.1.93}$$

Upon squaring both sides of (C.1.93) and using the bound $|\delta_r^{m'}| \leq 2C_{u_0}$ (a consequence of Theorem 5.4.1), we deduce that

$$\left(\delta_r^{n+1}\right)_+^2 \leq (\delta_r^{m'})_+^2 + \tilde{\Psi}\left(|\Delta k_{r+\frac{1}{2}}| + |\Delta k_{r-\frac{1}{2}}|\right), \tag{C.1.94}$$

where $\tilde{\Psi} := 16\lambda^2\|f_k\|^2\|k\| + 8C_{u_0}\lambda\|f_k\|$. Analogous arguments for $j = l-1$ yield

$$\left(\delta_{l-1}^{n+1}\right)_+^2 \leq (\delta_{l-1}^{m'})_+^2 + \tilde{\Psi}\left(|\Delta k_{l-\frac{1}{2}}| + |\Delta k_{l-\frac{3}{2}}|\right). \tag{C.1.95}$$

Thus, we observe that for any $j \in \mathbb{Z}$, either $(\delta_j^{n+1})_+^2 = (\delta_j^{m'})_+^2$ or $(\delta_j^{n+1})_+^2 \leq (\delta_j^{m'})_+^2 + \tilde{\Psi}(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|)$, for some $m \in S$. Therefore,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(\delta_j^{n+1}\right)_+^2 &\leq \sum_{m\in S}\sum_{j\in\Gamma_m}\left(\delta_j^{m'}\right)_+^2 + 2\tilde{\Psi}\|k\|_{BV} \\ &= \sum_{m\in S_\uparrow}\sum_{j\in\Gamma_m}\left(\delta_j^{m'}\right)_+^2 + \sum_{m\in S_\downarrow}\sum_{j\in\Gamma_m}\left(\delta_j^{m'}\right)_+^2 + 2\tilde{\Psi}\|k\|_{BV}\end{aligned} \tag{C.1.96}$$

where $S_\uparrow := \{m \in S : \{u_j^m\}_{j\in\mathbb{Z}}$ is a non-decreasing sequence$\}$ and $S_\downarrow := \{m \in S : \{u_j^m\}_{j\in\mathbb{Z}}$is a non-increasing sequence$\}$. Finally, invoking Lemma C.1.1 for $\{u_j^m\}_{j\in\mathbb{Z}}$, $m \in S_\uparrow$ and Lemma C.1.9 for $\{u_j^m\}_{j\in\mathbb{Z}}$, $m \in S_\downarrow$ in (C.1.96) and recollecting (C.1.90), we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(\delta_j^{n+1}\right)_+^2 \leq& \sum_{m\in S_\uparrow}\sum_{j\in\Gamma_m}(\delta_{j-\frac{1}{2}}^m)_+^2 - \frac{1}{500}\lambda\gamma_1\sum_{m\in S_\uparrow}\sum_{j\in\Gamma_m}(\delta_{j-\frac{1}{2}}^m)_+^3 \\ &- \frac{1}{6400}\sum_{m\in S_\uparrow}\sum_{j\in\Gamma_m}\left(\Delta^2\delta_{j-\frac{3}{2}}^m\right)^2 + 3\Theta\|k\|_{BV} \\ &+ \sum_{m\in S_\downarrow}\sum_{j\in\Gamma_m}\lambda^2\|f_k\|^2\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right)^2 + 2\tilde{\Psi}\|k\|_{BV} \\ \leq& \sum_{j\in\mathbb{Z}}(\delta_{j-\frac{1}{2}}^n)_+^2 - \frac{1}{500}\lambda\gamma_1\sum_{j\in\mathbb{Z}}(\delta_{j-\frac{1}{2}}^n)_+^3 + 3\Theta\|k\|_{BV} + 8\|k\|\lambda^2\|f_k\|^2\|k\|_{BV} \\ &+ 2\tilde{\Psi}\|k\|_{BV} \\ \leq& \sum_{j\in\mathbb{Z}}(\delta_{j-\frac{1}{2}}^n)_+^2 - \frac{1}{500}\lambda\gamma_1\sum_{j\in\mathbb{Z}}(\delta_{j-\frac{1}{2}}^n)_+^3 + \Psi\|k\|_{BV},\end{aligned} \tag{C.1.97}$$

where $\Psi := 3\Theta + 8\|k\|\lambda^2\|f_k\|^2 + 2\tilde{\Psi}$. Here, we have used the fact that

$$\sum_{j\in\mathbb{Z}}\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right)^2 \leq 4\|k\|\sum_{j\in\mathbb{Z}}\left(|\Delta k_{j+\frac{1}{2}}| + |\Delta k_{j-\frac{1}{2}}|\right) \leq 8\|k\|\|k\|_{BV}.$$

This completes the proof. $\square$

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Publications

Published work

1. G. D. Veerappa Gowda, K. Sudarshan Kumar and **N. Manoj.**, "Convergence of a second-order scheme for non-local conservation laws", `https://www.esaim-m2an.org/articles/m2an/abs/2023/06/m2an230129/m2an230129.html`, *ESAIM: M2AN*, 57 (6) 3439-3481(2023).

Manuscripts under review

1. **N. Manoj**, G. D. Veerappa Gowda and K. Sudarshan Kumar , "A positivity preserving second-order scheme for multi-dimensional system of non-local conservation laws"
   `https://arxiv.org/abs/2412.18475`

2. **N. Manoj** and K. Sudarshan Kumar , "Analysis of a central MUSCL-type scheme for conservation laws with discontinuous flux"
   `https://arxiv.org/abs/2501.04620`

3. **N. Manoj**, G. D. Veerappa Gowda and K. Sudarshan Kumar , "A MUSCL-Hancock scheme for non-local conservation laws"
   `https://www.arxiv.org/abs/2506.04176`